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THE CUBIC SZEGÖ EQUATION

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Résumé. On considère l'équation hamiltonienne suivante sur l'espace de Hardy du cercle

$$i\partial_t u = \Pi(|u|^2 u) ,$$

où Π désigne le projecteur de Szegö. Cette équation est un cas modèle d'équation sans aucune propriété dispersive. On établit qu'elle admet une paire de Lax et une infinité de lois de conservation en involution, et qu'elle peut être approchée par une suite de systèmes hamiltoniens de dimension finie complètement intégrables. Néanmoins, on met en évidence des phénomènes d'instabilité illustrant la dégénérescence de cette structure complètement intégrable. Enfin, on caractérise les ondes progressives de ce système.

Abstract. We consider the following Hamiltonian equation on the L^2 Hardy space on the circle,

$$i\partial_t u = \Pi(|u|^2 u) ,$$

where Π is the Szegö projector. This equation can be seen as a toy model for totally non dispersive evolution equations. We display a Lax pair structure for this equation. We prove that it admits an infinite sequence of conservation laws in involution, and that it can be approximated by a sequence of finite dimensional completely integrable Hamiltonian systems. We establish several instability phenomena illustrating the degeneracy of this completely integrable structure. We also classify the traveling waves for this system.

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1. Introduction

1.1. **Motivation.** This work can be seen as a continuation of a series of papers due to N. Burq, N. Tzvetkov and the first author [6, 7, 8, 9] — see also [12] for a survey— , devoted to the influence of the geometry of a Riemannian manifold M onto the qualitative properties of solutions to the nonlinear Schrödinger equation,

$$i\partial_t u + \Delta u = |u|^2 u \ , (t,x) \in \mathbb{R} \times M \ . \tag{1}$$

The usual strategy for finding global solutions to the Cauchy problem is to solve locally in time in the energy space $H^1 \cap L^4$ using a fixed point argument and then to globalize in time, by means of conservation of energy and of L^2 norm. As a corollary of the work of Burq, Gérard, Tzvetkov — see [8], remark 2.12 p.205, one obtains, whatever the geometry is, the following general result. If there exists a smooth local in time flow map on the Sobolev space $H^s(M)$, then the following Strichartz–type estimate must hold,

$$\|\mathrm{e}^{it\Delta} f\|_{L^4([0,1]\times M)} \lesssim \|f\|_{H^{s/2}(M)} \ . \tag{2}$$

This inequality is valid for instance if $M = \mathbb{R}^d, d = 1, 2, 3, 4$ and Δ is the Euclidean Laplacian, where s is given by the scaling formula

$$s = \max(0, \frac{d}{2} - 1) \ .$$

In [7, 8], it is observed that, on the two-dimensional sphere, the infimum of the numbers s such that (2) holds is $1/4$, hence is larger than the regularity given by the latter formula. This can be interpreted as a lack of dispersion properties for the spherical geometry. It is therefore natural to ask whether there exist some geometries for which these dispersion properties totally disappear. Such an example arises in sub-Riemannian geometry, more precisely for radial solutions of the Schrödinger equation associated to the sub-Laplacian on the Heisenberg group, as observed in [13], where part of the results of this paper are announced. Here we present a more elementary example of such a situation. Let us choose $M = \mathbb{R}^2_{x,y}$ and replace the Laplacian by the Grushin operator $G := \partial_x^2 + x^2\partial_y^2$, so that our equation is

$$i\partial_t u + \partial_x^2 u + x^2 \partial_y^2 u = |u|^2 u \ . \tag{3}$$

Notice that this equation enjoys the following scaling invariance : if $u(t, x, y)$ is a solution, then

$$\lambda u(\lambda^2 t, \lambda x, \lambda^2 y)$$

is also a solution. In this context it is natural to replace the standard Sobolev space $H^s(M)$ by the Grushin Sobolev space $H^s_G(M)$, defined as the domain of $\sqrt{(-G)^s}$. Observe that the above scaling transformation leaves invariant the homogeneous norm of $H_G^{1/2}(M)$, which suggests that equation (3) is *subcritical* with respect to the energy regularity

$H^1_G(M)$. However, we are going to see that (2) cannot hold if $s < \frac{3}{2}$, which means that no smooth flow can exist on the energy space, hence equation (3) should rather be regarded as *supercritical* with respect to the energy regularity. In fact, the critical regularity $s_c = \frac{3}{2}$ is the regularity which corresponds to the Sobolev embedding in M, since x has homogeneity 1 and y has homogeneity 2. This is an illustration of a total lack of dispersion for equation (3).

The justification is as follows. Notice that $u = e^{itG}f$ can be explicitly described by using the Fourier transform in the y variable, and by making an expansion along the Hermite functions h_m in the x variable, leading to the representation

$$u(t,x,y) = (2\pi)^{-1/2} \sum_{m=0}^{\infty} \int_{\mathbb{R}} \mathrm{e}^{-it(2m+1)|\eta| + iy\eta} \hat{f}_m(\eta) h_m(\sqrt{|\eta|}x)\, d\eta \ ,$$

with

$$\|f\|^2_{H_G^{s/2}} = \sum_{m=0}^{\infty} \int_{\mathbb{R}} (1 + (2m+1)|\eta|)^{s/2} |\hat{f}_m(\eta)|^2 \frac{d\eta}{\sqrt{|\eta|}}.$$

Let us focus onto data concentrated on modes $m = 0, \eta \sim N^2$, specifically

$$f(x,y) = \frac{1}{\sqrt{N}} \int_0^{\infty} \mathrm{e}^{iy\eta - \eta\frac{x^2}{2} - \frac{\eta}{N^2}}\, d\eta = N^{\frac{3}{2}} F(Nx, N^2 y)$$

with

$$F(x,y) := \frac{1}{1 + \frac{x^2}{2} - iy} \ .$$

Then the above formula for u gives

$$u(t,x,y) = f(x, y-t) \ ,$$

so that

$$\|u\|_{L^4([0,1]\times\mathbb{R}^2_{x,y})} = N^{3/4} \|F\|_{L^4} \ .$$

Since $\|f\|_{H_G^{s/2}} \simeq N^{s/2}$ as $N \to \infty$, this proves the claim.

Let us study the *structure of the nonlinear evolution problem (3)*. Denote by $V_m^{\pm}$ the space of functions of the form

$$v_m^{\pm}(x,y) = \int_0^{\infty} e^{\pm i\eta y} g(\eta) h_m(\sqrt{\eta}x)\, d\eta \ , \ \int_0^{\infty} \eta^{-1/2} |g(\eta)|^2\, d\eta < \infty \ ,$$

so that we have the orthogonal decomposition

$$L^2(M) = \oplus_{\pm} \oplus_{m=0}^{\infty} V_m^{\pm} \ , \ G_{|V_m^{\pm}} = \pm i(2m+1)\partial_y \ .$$

Denote by $\Pi_m^\pm : L^2(M) \to V_m^\pm$ the orthogonal projection. Expanding the solution as

$$u = \sum_{\pm} \sum_{m=0}^{\infty} u_m^\pm \ , \ u_m^\pm = \Pi_m^\pm u \ ,$$

the equation reads as a system of coupled transport equations,

$$i(\partial_t \pm (2m+1)\partial_y)u_m = \Pi_m^\pm(|u|^2 u) \ . \tag{4}$$

Therefore a better understanding of equation (3) requires to study the interaction between the nonlinearity $|u|^2u$ and the projectors $\Pi_m^\pm$. Notice that similar interactions arise in the literature, see for instance [22] in the study of the Lowest Landau Level for Bose-Einstein condensates, or [10] in the study of critical high frequency regimes of NLS on the sphere. Other examples can be found in the introduction of [13]. The present paper is devoted to a toy model for this kind of interaction.

1.2. **A toy model : the cubic Szegö equation.** Let

$$\mathbb{S}^1 = \{z \in \mathbb{C}, |z| = 1\}$$

be the unit circle in the complex plane. If u is a distribution on $\mathbb{S}^1$, $u \in \mathcal{D}'(\mathbb{S}^1)$, then u admits a Fourier expansion in the distributional sense

$$u = \sum_{k\in\mathbb{Z}} \hat{u}(k) e^{ik\theta} \ .$$

For every subspace E of $\mathcal{D}'(\mathbb{S}^1)$, we denote by E_+ the subspace

$$E_+ = \{u \in E \ ; \ \forall k < 0, \hat{u}(k) = 0\} \ .$$

In particular, L^2_+ is the Hardy space of L^2 functions which extend to the unit disc $\{|z| < 1\}$ as holomorphic functions,

$$u(z) = \sum_{k=0}^{\infty} \hat{u}(k) z^k \ , \ \sum_{k=0}^{\infty} |\hat{u}(k)|^2 < +\infty \ .$$

Let us endow $L^2(\mathbb{S}^1)$ with the scalar product

$$(u|v) := \int_{\mathbb{S}^1} u\overline{v} \, \frac{d\theta}{2\pi} \ ,$$

and denote by $\Pi : L^2(\mathbb{S}^1) \to L^2_+(\mathbb{S}^1)$ be the orthogonal projector on $L^2_+(\mathbb{S}^1)$, the so-called Szegö projector,

$$\Pi\left(\sum_{k\in\mathbb{Z}} \hat{u}(k) e^{ik\theta}\right) = \sum_{k\geq 0} \hat{u}(k) e^{ik\theta}.$$

We consider the following evolution equation on $L^2_+(\mathbb{S}^1)$,

$$i\partial_t u = \Pi(|u|^2 u) \ . \tag{5}$$

This equation, that we decided to call *the cubic Szegö equation*, is the simplest one which displays interaction between a cubic nonlinearity

and a Calderon-Zygmund projector. It is also an infinite dimensional Hamiltonian system on $L^2_+(\mathbb{S}^1)$, as we shall now see.

1.3. **The Hamiltonian formalism.** We endow $L^2_+(\mathbb{S}^1)$ with the symplectic form

$$\omega(u,v) = 4\,\mathrm{Im}(u|v)\ .$$

Given a real valued function F defined on a dense subspace $\mathcal{D}$ of $L^2_+(\mathbb{S}^1)$, we shall say that F admits a Hamiltonian vector field if there exists a mapping

$$X_F : \mathcal{D} \to L^2_+(\mathbb{S}^1)$$

such that, for every $h \in \mathcal{D}$,

$$\frac{F(u+th)-F(u)}{t} \underset{t\to 0}{\longrightarrow} \omega(h, X_F(u))\ .$$

Of course, this property is often strengthened as differentiability of F for some norm on $\mathcal{D}$ (see Kuksin [17] for a general setting in scales of Hilbert spaces). A Hamiltonian curve associated to F is a solution $u = u(t)$ of

$$\dot u = X_F(u)\ ,$$

and, given two functions F, G on $\mathcal{D}$ admitting Hamiltonian vector fields, the Poisson bracket of F, G is defined on $\mathcal{D}$ by

$$\{F, G\}(u) = \omega(X_F(u), X_G(u))\ .$$

For example, the function

$$E(u) = \int_{\mathbb{S}^1} |u|^4 \,\frac{d\theta}{2\pi}\ ,$$

defined on $L^4_+(\mathbb{S}^1)$, admits on $H^s_+(\mathbb{S}^1)$, $s > \frac{1}{2}$, the Hamiltonian vector field

$$X_E(u) = -i\Pi(|u|^2 u)\ ,$$

which defines a smooth vector field on H^s_+, so that equation (5) is the equation of Hamiltonian curves for E. From this structure, the equation (S) inherits the formal conservation law $E(u) = E(u(0))$. The invariance by translation and by multiplication by complex numbers of modulus 1 gives two other formal conservation laws,

$$Q(u) := \int_{\mathbb{S}^1} |u|^2 \,\frac{d\theta}{2\pi} = \|u\|^2_{L^2} \quad , \quad M(u) := (Du|u),\ D := -i\partial_\theta = z\partial_z\ .$$

Equivalently, these conservation laws mean that we have the following cancellations for the Poisson brackets,

$$\{E, Q\} = \{E, M\} = 0\ ,$$

which can be recovered in view of the explicit expressions of the Hamiltonian vector fields,

$$X_Q(u) = -\frac{i}{2}u \ , \ X_M(u) = -\frac{i}{2}Du \ .$$

Finally, these expressions also imply that

$$\{Q, M\} = 0 \ .$$

1.4. **Main results.** ¿From the previous conservation laws, we shall show — see section 2— that (5) defines a continuous flow on $H^{1/2}_+$. The main results of this paper are based on an unexpected property of this flow, namely that it admits a Lax pair, as the KdV flow (see Lax [18]) or the one dimensional cubic Schrödinger flow (see Zakharov-Shabat [31]). More precisely, for every $u \in H^{1/2}_+$, we define (see *e.g.* Peller [24], Nikolskii [21]), the Hankel operator of symbol u by

$$H_u(h) = \Pi(u\overline{h}) \ , \ h \in L^2_+ \ .$$

It is well known that H_u is a Hilbert-Schmidt operator, which is symmetric with respect to the real part of the scalar product on L^2_+. Our basic result is roughly the following — see section 3 for a more precise statement.

Theorem 1.1. *There exists a mapping $u \mapsto B_u$, valued into skew–symmetric operators on L^2_+, such that u is a solution of (5) if and only if*

$$\frac{d}{dt}H_u = [B_u, H_u] \ .$$

As a consequence, if u is a solution of (5), $H_{u(t)}$ is unitarily equivalent to $H_{u(0)}$. From this observation, we infer many new properties of the dynamics of (5), including an infinite sequence $(J_{2n})_{n\geq 1}$ of conservation laws in evolution. We also prove the approximation of equation (5) by finite dimensional completely integrable Hamiltonian systems — see sections 4 and 8.

Theorem 1.2. *For every positive integer D, there exists a complex submanifold $W(D)$ of $H^{1/2}_+$ of dimension D, such that*

1. *$W(D)$ is invariant by the flow of (5).*
2. *The flow of (5) is a completely integrable Hamiltonian flow on $W(D)$ in the Liouville sense.*

Moreover, the union of the manifolds $W(D)$, $D \geq 1$, is dense in $H^{1/2}_+$.

In Theorem 1.2 above, complete integrability in the Liouville sense means, according to Arnold [1], that for generic Cauchy data in $W(D)$, the evolution is quasi-periodic on a Lagrangian torus. In fact, $W(D)$

is a manifold of rational functions on the complex plane, with no poles in the unit disc. For instance, $W(3)$ consists of functions u given by

$$u(z) = \frac{az+b}{1-pz}$$

with $a \in \mathbb{C} \setminus \{0\}, b \in \mathbb{C}$, and p in the open unit disc. In this particular case, we solve (5) explicitly in section 6, and we deduce the following large time behavior of H^s norms of the solutions.

Theorem 1.3. *Every solution u of (5) on $W(3)$ satisfies*

$$\forall s > \frac{1}{2}\ ,\ \sup_{t\in\mathbb{R}} \|u(t)\|_{H^s} < +\infty.$$

However, there exists a family $(u_0^\varepsilon)_{\varepsilon>0}$ of Cauchy data in $W(3)$, which converges in $W(3)$ for the $C^\infty(\mathbb{S}^1)$ topology as $\varepsilon \to 0$, such that the corresponding solutions u^ε satisfy

$$\forall \varepsilon > 0, \exists t^\varepsilon > 0 : \forall s > \frac{1}{2}\ , \|u^\varepsilon(t^\varepsilon)\|_{H^s} \underset{\varepsilon\to 0}{\longrightarrow} +\infty.$$

The second statement of Theorem 1.3 is to be compared with the recent result by Colliander-Keel-Staffilani-Takaoka-Tao [11], who proved a similar behavior for cubic NLS on the two-dimensional torus. Notice that, as shown by our result, this behavior does not imply the existence of an unbounded trajectory in H^s, and that it can occur for completely integrable systems. This phenomenon shows that the conservation laws of equation (5) do not control the high energy Sobolev norms. However, let us mention that the boundedness of the trajectories in H^s is a generic property on all the manifolds $W(D)$, as we prove in section 7. The boundedness in H^s of the trajectory for all data in H^s for large s, is an interesting open problem.

Finally, in section 9 we characterize traveling waves for (5). In view of the two-dimensional symmetry group associated to Q and M, these traveling waves are defined as follows.

Definition 1. *A solution u of (5) is said to be a traveling wave if there exists $\omega, c \in \mathbb{R}$ such that*

$$u(t,z) = e^{-i\omega t}u(0, e^{-ict}z)$$

for every $t \in \mathbb{R}$. We shall call ω the pulsation of u, and c the velocity of u.

Notice that the equation for traveling waves is the following nonlinear equation,

$$cDu + \omega u = \Pi(|u|^2 u)\ .$$

In section 9, using the Lax pair structure and a precise spectral analysis of the corresponding selfadjoint operators, we describe all the solutions of this equation.

Theorem 1.4. *The initial data $u_0 \in H^{1/2}_+$ of traveling waves for (5) are given by*

$$u_0(z) = \begin{cases} \alpha \displaystyle\prod_{j=1}^{N} \frac{z - \overline{p}_j}{1 - p_j z} \text{ for } \alpha \in \mathbb{C}, |p_j| < 1, N \geq 1, \text{ if } c = 0 \ , \\ \alpha \dfrac{z^\ell}{1 - p^N z^N} \ , \text{ for } \alpha \in \mathbb{C}, N \geq 1, 0 \leq \ell \leq N-1 \ , \text{ if } c \neq 0. \end{cases}$$

The question of orbital stability of these traveling waves, in the sense of Grillakis-Shatah-Strauss [14], is of course very natural. We only have partial answers to this question, namely in the case $N = 1$ of the above theorem:

(1) For $|p| < 1$, the stationary wave corresponding to
$$u_0(z) = \frac{z - \overline{p}}{1 - pz}$$
is orbitally unstable — see section 6.

(2) For $|p| < 1$, the stationary wave corresponding to
$$u_0(z) = \frac{1}{1 - pz}$$
is orbitally stable — see section 5. In fact, we show that this data is a ground state of the variational equation which characterizes the traveling waves.

We close this introduction by mentioning two natural open problems, on which we hope to come back in a future work. The first one is to obtain a complete solution of equation (5) by solving inverse spectral problems for Hankel operators, describing explicit action angle coordinates for (5), as it is done in [15] for the KdV equation. The second one is of course to transfer at least part of the structure found here for attacking the open problem of global smooth solutions to the nonlinear Schrödinger equation associated to the Grushin operator, which was the starting point of this paper, and to other evolution problems on the same type [13], for instance on the Heisenberg group.

2. The Cauchy problem

In this section, we solve the Cauchy problem for the cubic Szegö equation, for sufficiently smooth data. We close the section by a remark about the smoothness of the flow map. Further results concerning uniform continuity of this flow map for weaker topologies can be found in Section 5, Proposition 6.

Theorem 2.1. *Given $u_0 \in H^{1/2}_+(\mathbb{S}^1)$, there exists a unique solution $u \in C(\mathbb{R}, H^{1/2}_+(\mathbb{S}^1))$ of (5) such that $u(0) = u_0$. For every $T > 0$, the mapping $u_0 \in H^{1/2}_+ \mapsto u \in C([-T,T], H^{1/2}_+)$ is continuous. Moreover, if $u_0 \in H^s_+(\mathbb{S}^1)$ for some $s > \frac{1}{2}$, then $u \in C(\mathbb{R}, H^s_+(\mathbb{S}^1))$.*

Proof. Assume first that $s > 1/2$. Since the vector field X_E is smooth on H^s_+, it is easy to solve (S) locally in time . More precisely, one has to solve the integral equation

$$u(t) = u_0 - i \int_0^t \Pi(|u|^2 u)\,dt'. \tag{6}$$

The corresponding operator is well defined on $H^s_+(\mathbb{S}^1)$ since

$$\|\Pi(|u|^2u)\|_{H^s} \le \||u|^2u\|_{H^s} \le C\|u\|^2_{L^\infty}\|u\|_{H^s} \le C'\|u\|^3_{H^s}.$$

This allows to use a fixed point argument on a small time interval, and yields a time interval of existence $[-T, T]$ where T is bounded from below if $\|u_0\|_{H^s}$ is bounded.

Next we show that the H^s-norm of this unique solution remains bounded on any time interval, so that this solution is global. To that purpose, we make use of the conservation of Q and M, and of the following observation,

$$M(u) + Q(u) = \sum_{k\ge 0}(k+1)|\hat{u}(k)|^2 = \|u\|^2_{H^{1/2}}\ . \tag{7}$$

So far, we have only observed that M and Q are formally conserved. In fact, it is straightforward to prove this conservation for sufficiently smooth solutions, and finally we get them for H^s solutions, $s > 1/2$, by approximation.

We combine the conservation of the $H^{1/2}$ norm with the following Brezis-Gallouët type estimate (see [5]),

$$\|u\|_{L^\infty} \le C_s\|u\|_{H^{1/2}}\left[\log\left(2 + \frac{\|u\|_{H^s}}{\|u\|_{H^{1/2}}}\right)\right]^{\frac{1}{2}}\ .$$

A proof of this estimate is recalled in Appendix 1. We infer, for $t \ge 0$,

$$\begin{aligned}\|u\|_{H^s} &\le \|u_0\|_{H^s} + \int_0^t \|\Pi(|u|^2u)\|_{H^s}dt' \le \|u_0\|_{H^s} + C\int_0^t \|u\|^2_{L^\infty}\|u\|_{H^s}dt' \\ &\le \|u_0\|_{H^s} + B\int_0^t \|u_0\|^2_{H^{1/2}}\left[\log\left(2 + \frac{\|u\|_{H^s}}{\|u_0\|_{H^{1/2}}}\right)\right]\|u\|_{H^s}dt'\ .\end{aligned}$$

If we set $f(t) := \|u\|_{H^s}/\|u_0\|_{H^{1/2}}$, we obtain

$$f(t) \le f(0) + A\int_0^t \left[\log(2 + f(t'))\right] f(t')dt'\ .$$

so that, by a non linear Gronwall lemma, f does not blow up in finite time ,

$$2 + f(t) \le (2 + f(0))^{\mathrm{e}^{At}}\ . \tag{8}$$

This completes the proof for $s > 1/2$.

Let us turn to the case $s = 1/2$. The proof of global existence of weak solutions is standard. Let us recall it briefly. Given $u_0 \in H^{1/2}_+$, approximate it by a sequence (u_0^n) of elements in H^s_+, $s > 1/2$. Consider the sequence (u_n) of solutions of (5) in $C(\mathbb{R}, H^s_+)$ corresponding to these initial data. In view of (7), the $H^{1/2}$ norm of $u_n(t)$ remains bounded for any $t \in \mathbb{R}$, and consequently $\partial_t u_n(t)$ remains bounded in, say, L^2. Hence there exists a subsequence of $u_n(t)$ converging weakly to $u(t)$ in $H^{1/2}$, locally uniformly in t. By the Rellich theorem, $u_n(t)$ converges strongly to $u(t)$ in L^p for every $p < \infty$, and it is easy to check that such a function u is a weak solution of (5).

Next, let us prove the uniqueness, which follows from an argument first introduced by Yudovich in the case of the $2D$ Euler equation and used by Vladimirov in [27], and Ogawa in [23]. It is based on the fact that functions in $H^{1/2}(\mathbb{S}^1)$ satisfy the Trudinger-type inequality,

$$\forall p \in [1, \infty[\,,\ \|u\|_{L^p} \le C\,\sqrt{p}\,\|u\|_{H^{1/2}} \tag{9}$$

We postpone the proof of this estimate to Appendix 2. Let u and $\tilde u$ be two solutions of (5) belonging to $C_w(\mathbb{R}, H^{1/2}_+)$ with $u(0) = \tilde u(0)$. Set $g(t) := \|u(t) - \tilde u(t)\|^2_{L^2}$ so that g is C^1 and vanishes at the origin. Introduce a large number $p > 2$ and compute

$$\begin{aligned}
|g'(t)| &= 2\,\left|\mathrm{Im}\left(\,(u(t)-\tilde u(t))\,|\,\Pi(|u|^2u - |\tilde u|^2\tilde u)\,\right)\right| \\
&\le C_1 \int_{\mathbb{S}^1} |u-\tilde u|^2(|u|^2+|\tilde u|^2)d\theta \\
&\le C_1' \int_{\mathbb{S}^1} |u-\tilde u|^{2(1-\frac1p)}(|u|^2+|\tilde u|^2)^{1+\frac1p}d\theta \\
&\le C_2\|u-\tilde u\|_{L^2}^{2(1-\frac1p)}(\|u\|_{L^{2(p+1)}}^{2(1+\frac1p)} + \|\tilde u\|_{L^{2(p+1)}}^{2(1+\frac1p)}) \\
&\le B\,p\,g(t)^{1-\frac1p}\ .
\end{aligned}$$

This implies

$$g(t) \le (Bt)^p\ .$$

The right hand side of the latter inequality goes to zero as p goes to infinity for any $t < 1/B$. This proves the uniqueness of the Cauchy problem.

It remains to prove that the weak solution u is strongly continuous in time with values in $H^{1/2}$, and that it depends continuously on the Cauchy data u_0. First, by weak convergence, we have $\|u(t)\|_{H^{1/2}} \le \|u_0\|_{H^{1/2}}$ for any $t \in \mathbb{R}$. By reversing time and using uniqueness, one obtains the converse inequality for any $t \in \mathbb{R}$ — solve the Cauchy problem with initial data $u(t)$. Hence the $H^{1/2}$ norm is preserved by the flow on $H^{1/2}_+$. Since u is weakly continuous with respect to t and since

$H^{1/2}_+$ is a Hilbert space, this completes the proof of the strong continuity of u. The continuity of the flow map can be proved similarly. $\square$

Remark 1. *For $s > 1/2$, the contraction mapping argument used to construct the solution u classically allows to prove that the flow map $u_0 \mapsto u(t)$ is Lipschitz continuous on bounded subsets of H^s and that it is smooth.*

On the opposite, the flow defined on $H^{1/2}_+(\mathbb{S}^1)$ is not smooth — in fact it is not C^3 near 0. Here is the argument. If Φ_t is the flow map, a simple expansion shows that, for $h \in H^s_+, s > \frac{1}{2}$,

$$d^3\Phi_t(0)(h,h,h) = -6it\Pi(|h|^2h) \ .$$

Hence the fact that Φ_1 is C^3 on a neighborhood of 0 in $H^{1/2}_+$ is in contradiction with the existence of $h \in H^{1/2}_+$ such that $\Pi(|h|^2h)$ does not belong to $H^{1/2}_+$. As a simple computation shows, an example of such a function h is given by $h_\alpha = f^\alpha$ where $f(z) = -\frac{\log(1-z)}{z}$ and $\frac{1}{6} < \alpha < \frac{1}{2}$.

3. A Lax pair for the cubic Szegö equation.

In this section, we show that the cubic Szegö equation (5) enjoys a very rich property, namely it admits a Lax pair in the sense of Lax [18]. As a preliminary step, we introduce relevant operators on the Hardy space $L^2_+(\mathbb{S}^1)$ (see Nikolskii [21] and Peller [24] for general references).

Given $u \in H^{1/2}_+(\mathbb{S}^1)$, the *Hankel operator* of symbol u is defined by

$$H_u(h) = \Pi(u\overline{h}) \ .$$

Notice that H_u is $\mathbb{C}$-antilinear, and is always a symmetric operator with respect to the real scalar product $\mathrm{Re}(u|v)$. In fact, it satisfies the identity

$$(H_u(h_1)|h_2) = (H_u(h_2)|h_1) \ .$$

Consequently, H_u^2 is $\mathbb{C}$-linear, selfadjoint and nonnegative. Moreover, H_u is given in terms of Fourier coefficients by

$$\widehat{H_u(h)}(k) = \sum_{\ell\ge 0} \hat{u}(k+\ell)\overline{\hat{h}(\ell)} \ .$$

Consequently, we have

$$\widehat{H_u^2(h)}(k) = \sum_{j\le 0} c_{kj}h_j \ , \ c_{kj} := \sum_{\ell \ge 0} \hat{u}(k+\ell)\overline{\hat{u}_(j+\ell)} \ .$$

In particular,

$$Tr(H_u^2) = \sum_{k\ge 0} c_{kk} = \sum_{\ell\ge 0}(\ell+1)|\hat{u}(\ell)|^2 = M(u) + Q(u) \ , \tag{10}$$

hence H_u is a Hilbert-Schmidt operator.

Given $b \in L^\infty(\mathbb{S}^1)$, the *Toeplitz operator* of symbol b is defined by

$$T_b(h) = \Pi(bh) \ .$$

The operator T_b is of course $\mathbb{C}$ -linear, and is selfadjoint for the Hermitian scalar product (hence symmetric for the real scalar product) as soon as b is real valued.

Theorem 3.1. *Let $u \in C(\mathbb{R}, H^s(\mathbb{S}^1))$ for some $s > \frac{1}{2}$. The cubic Szegö equation*

$$i\partial_t u = \Pi(|u|^2 u)$$

is equivalent to the fact that the Hankel operator H_u satisfies the evolution equation

$$\frac{d}{dt} H_u = [B_u, H_u] \tag{11}$$

where

$$B_u = \frac{i}{2} H_u^2 - iT_{|u|^2} \tag{12}$$

is a skew-symmetric operator. In other words, the pair (H_u, B_u) is a Lax pair for the cubic Szegö equation.

Proof. Firstly, we establish the following identity,

$$H_{\Pi(|u|^2 u)} = T_{|u|^2} H_u + H_u T_{|u|^2} - H_u^3. \tag{13}$$

Given $h \in L^2_+$, we have

$$H_{\Pi(|u|^2 u)}(h) = \Pi(\Pi(|u|^2 u)\overline{h}) = \Pi(|u|^2 u\overline{h})$$

since $\Pi((1-\Pi)(b)\overline{h}) = 0$ for every b. Then

$$\Pi(|u|^2 u\overline{h}) = \Pi(|u|^2 \Pi(u\overline{h})) + \Pi(|u|^2 (1-\Pi)(u\overline{h})),$$

and we observe that

$$\Pi(|u|^2 \Pi(u\overline{h})) = T_{|u|^2} H_u(h) \ ,$$

while

$$\Pi(|u|^2 (1-\Pi)(u\overline{h})) = H_u \left(u\overline{(1-\Pi)(u\overline{h})} \right) \ .$$

It remains to notice that, since $u\overline{(1-\Pi)(u\overline{h})} \in L^2_+$,

$$\begin{aligned} u\overline{(1-\Pi)(u\overline{h})} &= \Pi \left(u\overline{(1-\Pi)(u\overline{h})} \right) \\ &= \Pi(|u|^2 h) - \Pi \left(u\overline{\Pi(u\overline{h})} \right) = T_{|u|^2}(h) - H_u^2(h) \ . \end{aligned}$$

This completes the proof of (13). Now we just observe that (5) is equivalent to

$$\frac{d}{dt} H_u = -iH_{\Pi(|u|^2 u)} = [B_u, H_u]$$

since H_u is antilinear. $\square$

As a consequence of Theorem 3.1, the cubic Szegö equation admits an infinite number of conservation laws. Indeed, from (11), we classically observe that, denoting by $U(t)$ the solution of the operator equation

$$\frac{d}{dt}U = B_u U, U(0) = I \ ,$$

the operator $U(t)$ is unitary for every t, and

$$U(t)^* H_{u(t)} U(t) = H_{u(0)} \ .$$

In other words, we have the following property.

Corollary 1. *Let u be a solution of (5) with initial value $u_0 \in H^s_+, s > 1/2$. The family of Hankel operators $(H_{u(t)})_{t\in\mathbb{R}}$ is isospectral to H_{u_0}.*

Let us state some consequences of this isospectrality. First, we recall some basic properties of Hankel operators (see [21], [24] for proofs). It is well known from a theorem by Nehari [20] that the operator norm of H_u is equivalent to $\|u\|_{BMO} + \|u\|_{L^2}$, which is therefore essentially conserved by the flow. Moreover, a theorem by Peller states that, for $p < \infty$, the Schatten norm $[\mathrm{Tr}(|H_u|^p)]^{1/p}$ is equivalent to the norm of u in the Besov space $B^{1/p}_{p,p}$, which is therefore uniformly bounded for all time if it is finite at $t = 0$. Notice that the particular case $p = 2$ was already observed in (10), giving again the conservation of $M(u)+Q(u)$. Another example of a conserved quantity is of course the trace norm $Tr(|H_u|)$, which, as stated before, is equivalent to the Besov $B^1_{1,1}$ norm of u (or to the L^1-norm of u'' with respect to the area measure in the disc). This observation leads to a significant improvement of the large time estimate (8) for the high Sobolev norms of the solution of (5) derived from the proof of Theorem 2.1.

Corollary 2. *Assume $u_0 \in H^s_+$ for some $s > 1$. Then we have the following estimates,*

$$\begin{aligned} \sup_{t\in\mathbb{R}} \|u(t)\|_{L^\infty} &\leq C\|u_0\|_{H^s} \ , \\ \|u(t)\|_{H^s} &\leq C\|u_0\|_{H^s}\, \mathrm{e}^{C\|u_0\|_{H^s}|t|} \ . \end{aligned}$$

Proof. Since $H^s \subset B^1_{1,1}$ as soon as $s > 1$, the trace norm of H_{u_0} is finite, hence the $B^1_{1,1}$ norm of $u(t)$ is uniformly bounded. Since $B^1_{1,1} \subset L^\infty$, this proves the first assertion. The second one is then a simple consequence of the standard Gronwall lemma. □

We will return to the large time behavior of solutions of (5) in sections 6 and 7. At this stage, it is natural to find a way to recover other known conservation laws, namely Q and E. In fact, we are going to find them as two particular cases of an infinite sequence of conservation laws, which will play an important role in the sequel.

Corollary 3. *For every $u \in H^{1/2}_+$, for every positive integer n, set*

$$J_n(u) = (H_u^n(1)|1)\ .$$

If $u \in C(\mathbb{R}, H^{1/2}_+)$ solves (5), we have, for every positive integer k,

$$\frac{d}{dt}J_{2k}(u) = 0\ ,\ i\frac{d}{dt}J_{2k-1}(u) = J_{2k+1}(u)\ .$$

Proof. We may assume that $u_0 \in H^s$ for $s > 1/2$, since the general case follows by density and the continuity properties of the flow map on $H^{1/2}_+$. Coming back to (11), we observe that

$$B_u(1) = \frac{i}{2}H_u^2(1) - iT_{|u|^2}(1) = -\frac{i}{2}H_u^2(1)\ .$$

Consequently, since H_u^{2k} is $\mathbb{C}$-linear and B_u is skew symmetric,

$$\begin{aligned}
\frac{d}{dt}(H_u^{2k}(1)|1) &= ([B_u, H_u^{2k}](1), |1) \\
&= -(H_u^{2k}(1)|B_u(1)) - (H_u^{2k}B_u(1)|1) \\
&= -\frac{i}{2}(H_u^{2k+2}(1)|1) + \frac{i}{2}(H_u^{2k+2}(1)|1) = 0\ .
\end{aligned}$$

The second identity is obtained similarly, observing that H_u^{2k-1} is $\mathbb{C}$ -antilinear,

$$\begin{aligned}
i\frac{d}{dt}(H_u^{2k-1}(1)|1) &= i([B_u, H_u^{2k-1}](1), |1) \\
&= -i(H_u^{2k-1}(1)|B_u(1)) - i(H_u^{2k-1}B_u(1)|1) \\
&= \frac{1}{2}(H_u^{2k+1}(1)|1) + \frac{1}{2}(H_u^{2k+1}(1)|1) = J_{2k+1}(u)\ .
\end{aligned}$$

□

The conservation of Q and E is recovered by observing that

$$\begin{aligned}
J_2(u) &= (H_u^2(1)|1)_{L^2} = \|u\|_{L^2}^2 = Q(u)\ , \\
J_4(u) &= (H_u^4(1)|1)_{L^2} = \|H_u^2(1)\|_{L^2}^2 = \|\Pi(|u|^2)\|_{L^2}^2 = \frac{E(u) + Q(u)^2}{2}\ .
\end{aligned}$$

In section 8, we will prove that the conservation laws J_{2k} are in involution, and that that their differentials satisfy some generic independence.

4. Invariant finite dimensional submanifolds

In this section, we introduce finite dimensional submanifolds of L^2_+ which are invariant by the flow of the cubic Szegö equation. Elements of these manifolds turn out to be rational functions of the variable z, with no poles in the unit disc. In what follows, $\mathbb{C}_D[z]$ denotes the class of complex polynomials of degree at most D, and $d(A)$ denotes the degree of a polynomial A.

4.1. **The manifold $\mathcal{M}(N)$.**

Definition 2. *Let N be a positive integer. We denote by $\mathcal{M}(N)$ the set of rational functions u of the form*

$$u(z) = \frac{A(z)}{B(z)} ,$$

with $A \in \mathbb{C}_{N-1}[z]$, $B \in \mathbb{C}_N[z]$, $B(0) = 1$, $d(A) = N - 1$ or $d(B) = N$, A and B have no common factors, and $B(z) \neq 0$ if $|z| \leq 1$.

Notice that $\mathcal{M}(N)$ is included in H^s_+ for every s. It is elementary to check that $\mathcal{M}(N)$ is a $2N$-dimensional complex submanifold of L^2_+, and that its tangent space at $u = A/B$ is

$$T_u\mathcal{M}(N) = \frac{\mathbb{C}_{2N-1}[z]}{B^2} .$$

A theorem by Kronecker states that $\mathcal{M}(N)$ is exactly the set of symbols u such that H_u is of rank N. For the convenience of the reader, we give an elementary proof of this result in Appendix 3. In view of Corollary 1, we infer the following result, which can also be checked directly, using some elementary linear algebra.

Theorem 4.1. *Let $u_0 \in \mathcal{M}(N)$ and u be the solution of (5) with $u(0) = u_0$. Then, for every $t \in \mathbb{R}$, $u(t)$ belongs to $\mathcal{M}(N)$. In other words, the submanifolds $\mathcal{M}(N)$ are invariant under the flow of the cubic Szegö equation.*

In the notation of Theorem 1.2 of the introduction, the manifold $\mathcal{M}(N)$ is $W(2N)$. Since $\mathcal{M}(N)$ is finite dimensional, equation (5) on $\mathcal{M}(N)$ is reduced to a system of ordinary differential equations, which we now describe in the main coordinate patch of $\mathcal{M}(N)$. A generic point in $\mathcal{M}(N)$ is given by

$$u = \sum_{j=1}^{N} \frac{\alpha_j}{1 - p_j z},$$

where the p_j's are pairwise distinct and belong to the unit disc. Then, in the coordinates $(\alpha_j, p_j)_{1\leq j\leq N}$, (5) reads

$$\begin{cases} i\dot{\alpha}_j &= \sum_k \frac{\alpha_j^2\overline{\alpha_k}}{(1-p_j\overline{p_k})^2} + 2\sum_k \sum_{\ell\neq j} \frac{\alpha_j\overline{\alpha_k}\alpha_\ell p_j}{(p_j-p_\ell)(1-p_j\overline{p_k})} , \\ i\dot{p}_j &= \sum_k \frac{\alpha_j\overline{\alpha_k}}{1-p_j\overline{p_k}}\, p_j , \end{cases} \tag{14}$$

In particular, the conservation laws Q, M, E read

$$\begin{aligned} Q &= \sum_{j,k} \frac{\alpha_j\overline{\alpha}_k}{1 - p_j\overline{p}_k} , \ M = \sum_{j,k} \frac{\alpha_j p_j\overline{\alpha}_k\overline{p}_k}{(1 - p_j\overline{p}_k)^2} , \\ E &= \sum_{j,k,l,m} \frac{\alpha_j\overline{\alpha}_k\alpha_l\overline{\alpha}_m(1 - p_j\overline{p}_k p_l\overline{p}_m)}{(1 - p_j\overline{p}_k)(1 - p_j\overline{p}_m)(1 - p_l\overline{p}_k)(1 - p_l\overline{p}_m)} . \end{aligned}$$

In view of the second part of system (14), we notice an additional conservation law,

$$S = |p_1 \cdots p_N|^2 \ . \tag{15}$$

In the next subsection, we give an intrinsic interpretation of S and we establish further properties which will be useful in the sequel.

4.2. **The Blaschke product associated to $u \in \mathcal{M}(N)$.** Given $u \in H^{1/2}_+$, it is elementary to check from

$$H_u(h) = \Pi(u\overline{h})$$

that $\ker H_u$ is a closed subspace of H_u invariant by the shift $h \mapsto zh$. According to the Beurling Theorem [25], there exists $\varphi \in L^2_+$, such that $|\varphi|^2 = 1$ on $\mathbb{S}^1$ and

$$\ker H_u = \varphi L^2_+ \ .$$

Let us characterize such a generator φ if $u = A/B \in \mathcal{M}(N)$. Set

$$B(z) = \prod_{j=1}^{N} (1 - p_j z) \ ,$$

where the p_j's are complex numbers in the open unit disc, with possible repetitions. We define the *Blaschke product* associated to u by

$$b(z) = \prod_{j=1}^{N} \frac{z - \overline{p}_j}{1 - p_j z} \ .$$

and we claim that

$$\ker H_u = bL^2_+ \ . \tag{16}$$

Indeed, if $u = A/B$ as in definition 4, the equation $\Pi(u\overline{h}) = 0$ means exactly that there exists $g \in L^2_+$ such that

$$z^{N-1}\overline{A}\left(\frac{1}{z}\right) h(z) = g(z) \prod_{j=1}^{N} (z - \overline{p}_j) \ .$$

On the other hand, the assumptions on A, B imply that the polynomials $z^{N-1}\overline{A}\left(\frac{1}{z}\right)$ and $\prod_{j=1}^{N}(z - \overline{p}_j)$ have no common factor. Consequently, $\ker H_u$ consists of those $h \in L^2_+$ which are divisible by $\prod_{j=1}^{N}(z - \overline{p}_j)$, which is equivalent to $h \in bL^2_+$.

Let us make the connection with the distinguished vector 1. Since $\mathrm{Im}(H_u)$ is finite dimensional and since H_u is symmetric, we have

$$\mathrm{Im}(H_u) = (\ker(H_u))^{\perp} \ .$$

In particular, $\mathrm{Im}(H_u)$ is a space of rational functions, whose general description is provided in Appendix 3. We denote by P_u the orthogonal projector on $\mathrm{Im}(H_u)$.

Proposition 1. *We have*

$$1 - P_u(1) = (-1)^N p_1 \cdots p_N \, b \ .$$

Proof. Set $v = 1 - P_u(1)$. From Appendix 3, $1 \in \mathrm{Im}(H_u)$ if and only if one of the p_j's is 0. Since the claimed identity is trivial in this case, we may assume that $p_j \neq 0$ for every j. Then, from Appendix 3, all the functions in $\mathrm{Im}(H_u)$ tend to 0 at infinity, hence $v(z)$ tends to 1 at infinity. Since

$$v(z) = h(z) b(z) \ ,$$

where h is a polynomial, we conclude that

$$h(z) = (-1)^N p_1 \cdots p_N \ .$$

□

As a consequence, we obtain the following interpretation of the conservation law S introduced in the previous subsection,

$$S := |p_1 \cdots p_N|^2 = \mathrm{dist}(u, \ker H_u)^2 \ .$$

Indeed, $\mathrm{dist}(u, \ker H_u)^2 = \|1 - P_u(1)\|^2_{L^2}$ and $|b|^2 = 1$ on $\mathbb{S}^1$, hence we even have $|1 - P_u(1)|^2 = S$ on $\mathbb{S}^1$. In fact, we can derive a more general evolution law for the whole quantity $v = 1 - P_u(1)$.

Proposition 2. *Let u be a solution of (5) on $\mathcal{M}(N)$. Then $v := 1 - P_u(1)$ satisfies, on $\mathbb{S}^1$,*

$$i\partial_t v = |u|^2 v \ .$$

Proof. Notice that v is the orthogonal projection of 1 onto $\ker H_u$, which reads, in terms of the functional calculus of the selfadjoint operator H_u^2,

$$v = \mathbf{1}_{\{0\}}(H_u^2)(1) \ .$$

Consequently, by Theorem 3.1,

$$\partial_t v = [B_u, \mathbf{1}_{\{0\}}(H_u^2)](1) \ .$$

Since

$$B_u = -iT_{|u|^2} + \frac{i}{2} H_u^2 \ , \ B_u(1) = -\frac{i}{2} H_u^2(1) \ ,$$

we get

$$i\partial_t v = T_{|u|^2} v \ .$$

The following lemma implies that $T_{|u|^2} v = |u|^2 v$ and therefore completes the proof.

Lemma 1. *If $u \in L^\infty_+ \cap H^{1/2}_+$ and $h \in \ker H_u$, then $\overline{u} h \in z L^2_+$.*

Indeed, for every $k \geq 0$, we have , in $L^2(\mathbb{S}^1)$,

$$(\overline{u} h | \overline{z}^k) = (z^k | u \overline{h}) = (z^k | \Pi(u \overline{h})) = (z^k | H_u(h)) = 0 \ .$$

□

As a consequence of the above proposition, let us deduce an evolution law for the Blaschke product b if $S \neq 0$. In this case, v does not vanish on the circle, and we can write, at each point of $\mathbb{S}^1$,

$$|u|^2 = i\frac{\partial_t v}{v} = i\frac{\partial_t (p_1 \cdots p_N)}{p_1 \cdots p_N} + i\frac{\partial_t b}{b} \ .$$

Let us take the average of both sides on $\mathbb{S}^1$. Since

$$\frac{\partial_t b}{b} = \sum_{j=1}^{N} \left(-\frac{\partial_t \overline{p}_j}{z - \overline{p}_j} + \frac{z \partial_t p_j}{1 - p_j z} \right) \ ,$$

a direct calculation yields

$$\int_{\mathbb{S}^1} \frac{\partial_t b}{b} \frac{dz}{2i\pi z} = 0 \ ,$$

and therefore

$$Q = i\frac{\partial_t (p_1 \cdots p_N)}{p_1 \cdots p_N} \ . \tag{17}$$

Coming back to Proposition 2, we infer

$$i\partial_t b = (|u|^2 - Q)b \ . \tag{18}$$

Equation (18) in fact holds without assuming $S \neq 0$. This can be shown by approximation in $\mathcal{M}(N)$. However, we shall give a different proof in the next subsection, which is devoted to the flow on the subset $\{S = 0\}$ of $\mathcal{M}(N)$.

4.3. **The manifold $\tilde{\mathcal{M}}(N-1)$.** Denote by $\tilde{\mathcal{M}}(N-1)$ the subset of $\mathcal{M}(N)$ defined by the equation $S = 0$. The rational functions in $\tilde{\mathcal{M}}(N-1)$ are the elements of $\mathcal{M}(N)$ with a numerator of degree exactly equal to $N-1$ and a denominator of degree at most $N-1$, therefore $\tilde{\mathcal{M}}(N-1)$ is a complex hypersurface of $\mathcal{M}(N)$, and its tangent space at $u = A/B$ is

$$T_u \tilde{\mathcal{M}}(N-1) = \frac{\mathbb{C}_{2N-2}[z]}{B^2} \ .$$

As S is invariant under the flow, we get that $\tilde{\mathcal{M}}(N-1)$ is invariant under the flow. In the notation of Theorem 1.2 of the introduction, $\tilde{\mathcal{M}}(N-1)$ is $W(2N-1)$. On this submanifold, generic points are described as

$$u = \sum_{j=1}^{N-1} \frac{\alpha_j}{1 - p_j z} + \alpha_N \ ,$$

where the p_j's are as before in the open unit disc, pairwise distincts and different from 0. The generic evolution is system (14) with $p_N = 0$.

From this explicit system, we notice that the trivial conservation law S is replaced by

$$\tilde{S} = \left|\alpha_N \prod_{j=1}^{N-1} p_j\right|^2 \ .$$

As in the previous section, we shall now give a more intrinsic interpretation of the new conservation law $\tilde{S}$.

Since $1 \in \mathrm{Im}(H_u) = \mathrm{Im}(H_u^2)$, there exists a unique $w \in \mathrm{Im}(H_u)$ such that

$$H_u(w) = 1 \ .$$

Write

$$u = \frac{A}{B} \ , \ B(z) = \prod_{j=1}^{N-1}(1 - p_j z) \ , \ A(z) = az^{N-1} + \sum_{j<N-1} a_j z^j \ ,$$

with $a \neq 0$. The associated Blaschke product now reads

$$b(z) = z \prod_{j=1}^{N-1} \frac{z - \overline{p}_j}{1 - p_j z} := z\tilde{b}(z) \ .$$

Notice that, from the description of $\mathrm{Im}(H_u)$ provided in Appendix 3, $\tilde{b} \in \mathrm{Im}(H_u)$. From the elementary identity

$$H_u(zh) = \overline{z}(H_u(h) - (u|h)) \ ,$$

we infer

$$H_u(\tilde{b}) = (u|\tilde{b}) \ .$$

Then an explicit calculation gives

$$(\tilde{b}|u) = \int_{\mathbb{S}^1} \frac{z^{N-1}\overline{A}(1/z)}{\prod_j (1 - p_j z)} \frac{dz}{2i\pi z} = \overline{a} \ .$$

Therefore we have proved

Proposition 3.

$$w(z) = \frac{\tilde{b}(z)}{\overline{a}} = \frac{b(z)}{\overline{a}z} \ .$$

We conclude this subsection by deriving an evolution law for w.

Proposition 4. *Let u be a solution of (5) on $\tilde{\mathcal{M}}(N-1)$. Then the preimage w of* 1 *in* $\mathrm{Im}(H_u)$ *satisfies, on* $\mathbb{S}^1$,

$$i\partial_t w = |u|^2 w \ .$$

Proof. The proof is very similar to the one of Proposition 2. Firstly, we express w by means of the functional calculus of the selfadjoint operator H_u^2,

$$w = f(H_u^2)H_u(1) \ , \ f(\lambda) := \frac{\mathbf{1}_{]0,\infty[}(\lambda)}{\lambda} \ .$$

Consequently, by Theorem 3.1,

$$\partial_t w = [B_u, f(H_u^2)H_u](1) \ .$$

Since

$$B_u = -iT_{|u|^2} + \frac{i}{2}H_u^2 \ , \ B_u(1) = -\frac{i}{2}H_u^2(1) \ ,$$

we get

$$i\partial_t w = T_{|u|^2} w \ .$$

The following lemma implies that $T_{|u|^2} w = |u|^2 w$ and therefore completes the proof.

Lemma 2. *If $u \in \tilde{\mathcal{M}}(N-1)$, then $\overline{u}w \in L^2_+$.*

Indeed, for every $k \geq 1$, we have , in $L^2(\mathbb{S}^1)$,

$$(\overline{u}w|\overline{z}^k) = (z^k|u\overline{w}) = (z^k|\Pi(u\overline{w})) = (z^k|H_u(w)) = (z^k|1) = 0 \ .$$

$\square$

As a consequence of Proposition 4, we infer that $\|w\|_{L^2}^2$ is a conservation law. In the case of a generic element of $\mathcal{M}(N-1)$,

$$u = \sum_{j=1}^{N-1} \frac{\alpha_j}{1 - p_j z} + \alpha_N \ ,$$

we have

$$a = (-1)^{N-1} p_1 \cdots p_{N-1} \alpha_N \ ,$$

thus we get the interpretation of $\tilde{S}$ as

$$\tilde{S} = |a|^2 = \frac{1}{\|w\|_{L^2}^2} \ .$$

Finally, as in the previous subsection, Proposition 4 leads to an evolution law for the coefficient a itself and for b. Indeed, taking the average on $\mathbb{S}^1$ of

$$|u|^2 = i\frac{\partial_t w}{w} = -i\frac{\partial_t(\overline{a})}{\overline{a}} + i\frac{\partial_t \tilde{b}}{\tilde{b}} = -i\frac{\partial_t(\overline{a})}{\overline{a}} + i\frac{\partial_t b}{b} \ ,$$

we obtain

$$i\partial_t a = Qa \ , \tag{19}$$

and, coming back to the equation on w, we eventually deduce the evolution of b (18), in the whole generality.

In the next two sections, we study the particular cases of $\mathcal{M}(1)$ and of $\tilde{\mathcal{M}}(1)$ in more detail.

5. The case of $\mathcal{M}(1)$

Elements of $\mathcal{M}(1)$ are

$$\varphi_{\alpha,p}(z) = \frac{\alpha}{1-pz} \ , \ \alpha \neq 0 \ , \ |p| < 1 \ . \tag{20}$$

In this particular case, the system (14) reads

$$i\dot{\alpha} = \frac{|\alpha|^2}{(1-|p|^2)^2}\alpha \ , \ i\dot{p} = \frac{|\alpha|^2}{1-|p|^2}p \ ,$$

which is solved as

$$\alpha(t) = \alpha(0)\,\mathrm{e}^{-i\omega t} \ , \ p(t) = p(0)\,\mathrm{e}^{-ict} \ , \ \omega = \frac{|\alpha(0)|^2}{(1-|p(0)|^2)^2} \ c = \frac{|\alpha(0)|^2}{1-|p(0)|^2} \ .$$

Equivalently, the solution u of (5) with $u(0) = \varphi_{\alpha,p}$ is given by

$$u(t,z) = \mathrm{e}^{-i\omega t}\,\varphi_{\alpha,p}(\mathrm{e}^{-ict}z),$$

which means that u is a traveling wave according to Definition 1. In section 9, we will classify all such solutions. Notice that, apart from the trivial case of constants — $p = 0$ —, the trajectory lies in the two-dimensional torus $\{|\alpha| = cst \ , \ |p| = cst\}$. We are going to prove that this two-dimensional torus can also be seen as the solution of a variational problem in $H^{1/2}_+$. We first state the following lemma which is an easy consequence of the Cauchy-Schwarz inequality.

Lemma 3. *Let A be a positive operator on a separable Hilbert space $\mathcal{H}$ and e be an element of $\mathcal{H}$ so that $Ae \neq 0$. Then, the following inequality holds*

$$\|Ae\|^2 \leq (Ae|e)\,Tr(A).$$

Furthermore, equality holds if and only if A is of rank one.

Applying this lemma to $A = H_u^2$ on $\mathcal{H} = L^2_+$ with $e = 1$, and using the formulae for J_2 and J_4 derived in section 3, we get the following characterization of the elements of $\mathcal{M}(1)$, which can be seen as an analogue of M. Weinstein's sharp Gagliardo-Nirenberg inequality [28].

Proposition 5. *For every $u \in H^{1/2}_+$,*

$$E(u) \leq Q(u)(Q(u) + 2M(u)),$$

i.e.

$$\|u\|^4_{L^4} \leq \|u\|^2_{L^2}(\|u\|^2_{L^2} + \|u\|^2_{H^{1/2}}) \ ,$$

with equality if and only if $u \in \mathcal{M}(1)$.

Let us mention that a more direct proof of Proposition 5 can be found in [13]. As a consequence of Proposition 5, we obtain the following large time stability of $\mathcal{M}(1)$ in $H^{1/2}_+$.

Corollary 4. *Let $a > 0$, $0 < r < 1$, and*

$$T(a,r) = \{\varphi_{\alpha,p} : |\alpha| = a \ , \ |p| = r\} \ .$$

For every $\varepsilon > 0$, there exists $\delta > 0$ such that, if $u_0 \in H^{1/2}_+$ satisfies

$$\inf_{\varphi \in T(a,r)} \|u_0 - \varphi\|_{H^{1/2}} \leq \delta$$

then the solution u of (5) with $u(0) = u_0$ satisfies

$$\sup_{t\in\mathbb{R}} \inf_{\varphi \in T(a,r)} \|u(t) - \varphi\|_{H^{1/2}} \leq \varepsilon \ .$$

Proof. By Proposition 5 and a simple calculation of $Q(\varphi_{\alpha,p})$ and $E(\varphi_{\alpha,p})$, $T(a,r)$ is the set of minimizers of the problem

$$\inf\{M(u) \ , \ u \in H^{1/2}_+ \ , \ Q(u) = q(a,r) \ , \ E(u) = e(a,r)\} = m(a,r) \ ,$$

where

$$q(a,r) := \frac{a^2}{1-r^2} \ , \ e(a,r) := \frac{a^4(1+r^2)}{(1-r^2)^3} \ .$$

Let u_0^n be a sequence of $H^{1/2}_+$ such that

$$\inf_{\varphi \in T(a,r)} \|u_0^n - \varphi\|_{H^{1/2}} \to 0 \ .$$

Then

$$Q(u_0^n) \to q(a,r) \ , \ E(u_0^n) \to e(a,r) \ , \ M(u_0^n) \to m(a,r)$$

and by the conservation laws,

$$Q(u^n(t)) \to q(a,r) \ , \ E(u^n(t)) \to e(a,r) \ , \ M(u^n(t)) \to m(a,r)$$

uniformly in t. Given any sequence (t_n) of real numbers, the sequence $(u^n(t_n))$ is bounded in $H^{1/2}_+$, hence has a subsequence which converges weakly to some u in $H^{1/2}_+$, and we get, by the weak continuity of Q, E and the weak semi-continuity of M,

$$Q(u) = q(a,r) \ , \ E(u) = e(a,r) \ , \ M(u) \leq m(a,r),$$

hence finally $M(u) = m(a,r)$, which implies from Proposition 5 that $u \in T(a,r)$ and that $u^n(t_n)$ converges strongly to u. The proof is complete. □

The explicit evolution of (5) on $\mathcal{M}(1)$ also allows to prove the following high frequency instability result in H^s_+ for every $s < 1/2$. This result means that, given a time $t \neq 0$, the flow map at time t does not extend as a uniformly continuous map on bounded subsets of H^s_+, $s < \frac{1}{2}$, or L^4_+ (see Tzvetkov [26] for a general discussion).

Proposition 6. *Let $s < \frac{1}{2}$. There exist u_0^ε, $\tilde{u}_0^\varepsilon$ bounded sequences in H^s_+ such that*

$$\|u_0^\varepsilon - \tilde{u}_0^\varepsilon\|_{H^s} \to 0 \text{ but } \forall t \neq 0, \ \liminf_{\varepsilon\to 0} \|u^\varepsilon(t) - \tilde{u}^\varepsilon(t)\|_{H^s} > 0 \ .$$

The same holds for H^s_+ replaced by L^4_+.

Proof. The principle of the proof follows Birnir-Kenig-Ponce-Svansted-Vega [3]. As $|p| \to 1$, one has

$$\|\varphi_{\alpha,p}\|_{H^s}^2 = \left\|\frac{\alpha}{1-pz}\right\|_{H^s}^2 \sim \frac{|\alpha|^2}{(1-|p|^2)^{1+2s}} \ .$$

Choose

$$u_0^\varepsilon = \varphi_{\varepsilon^{s+\frac12},\sqrt{1-\varepsilon}} \ , \ \tilde u_0^\varepsilon = \varphi_{\varepsilon^{s+\frac12}(1+\delta),\sqrt{1-\varepsilon}} \ ,$$

with $\delta \to 0$ so that $\|u_0^\varepsilon - \tilde u_0^\varepsilon\|_{H^s} \to 0$. By the previous computations, we get $u(t, \mathrm{e}^{i\theta}) = \mathrm{e}^{-i\omega t} u_0(\mathrm{e}^{i(\theta - ct)})$ and $\tilde u(t, \mathrm{e}^{i\theta}) = \mathrm{e}^{-i\tilde\omega t}\tilde u_0(\mathrm{e}^{i(\theta-\tilde c t)})$ where $\tilde c - c = \varepsilon^{2s}\delta(2+\delta)$. Choose $\varepsilon \to 0$ so that $\delta\varepsilon^{2s-1} \to \infty$. It implies in particular that $\dfrac{\tilde c - c}{\varepsilon} \to \infty$.

We claim that, for any $t > 0$,

$$\|u^\varepsilon(t) - \tilde u^\varepsilon(t)\|_{H^s}^2 = \|u^\varepsilon(t)\|_{H^s}^2 + \|\tilde u^\varepsilon(t)\|_{H^s}^2 + o(1)$$

as ε goes to zero. In other words, the scalar product in H^s of $u^\varepsilon(t)$ and $\tilde u^\varepsilon(t)$ is $o(1)$. The result will follow since $\|u^\varepsilon(t)\|_{H^s} \simeq \|\tilde u^\varepsilon(t)\|_{H^s} \simeq 1$.

We have

$$\begin{aligned}
|\langle u^\varepsilon(t), \tilde u^\varepsilon(t)\rangle_{H^s}| &= |\sum_k (1+|k|^2)^s \widehat{u^\varepsilon}(t,k) \cdot \overline{\widehat{\tilde u^\varepsilon}(t,k)}| \\
&= |\sum_k (1+|k|^2)^s e^{-ik(c-\tilde c)t} \widehat{u_0^\varepsilon}(k) \cdot \overline{\widehat{\tilde u_0^\varepsilon}(k)}| \\
&= \varepsilon^{2s+1}(1+\delta)|\sum_k (1+|k|^2)^s e^{-ik(c-\tilde c)t}(1-\varepsilon)^k| \\
&\simeq \frac{\varepsilon^{2s+1}}{|1-(1-\varepsilon)e^{-i(c-\tilde c)t}|^{1+2s}} \\
&\simeq \left(\frac{\varepsilon}{|c-\tilde c|t}\right)^{1+2s} = o(1)t^{-(1+2s)} \ .
\end{aligned}$$

The proof for L^4_+ is similar, observing that

$$\|\varphi_{\alpha,p}\|_{L^4}^4 = \frac{|\alpha|^4(1+|p|^2)}{(1-|p|^2)^3} \ .$$

Choose the same functions u_0^ε and $\tilde u_0^\varepsilon$ as above, with $s = \frac14$, and δ going to 0 such that $\delta\varepsilon^{-1/2} \to \infty$. In view of the explicit expression,

$$|u_0^\varepsilon(e^{i\theta})|^4 = \frac{\varepsilon^3}{(2-\varepsilon-2\sqrt{1-\varepsilon}\cos\theta)^2} \ ,$$

one easily checks that, if $R^\varepsilon \to \infty$,

$$\int_{R^\varepsilon\varepsilon<|\theta-ct|<\pi} |u^\varepsilon(t,e^{i\theta})|^4\, d\theta \to 0 \ , \qquad \int_{R^\varepsilon\varepsilon<|\theta-\tilde ct|<\pi} |\tilde u^\varepsilon(t,e^{i\theta})|^4\, d\theta \to 0 \ .$$

Let us choose R^ε such that

$$R^\varepsilon << \frac{\tilde{c} - c}{\varepsilon} \ .$$

Then we claim that, for $t \neq 0$,

$$\|u^\varepsilon(t) - \tilde{u}^\varepsilon(t)\|_{L^4}^4 = \|u^\varepsilon(t)\|_{L^4}^4 + \|\tilde{u}^\varepsilon(t)\|_{L^4}^4 + o(1) \ .$$

Indeed, if $a + b = 4$ and $a, b \geq 1$, by Hölder inequality, we have

$$\int\limits_{\mathbb{S}^1} |u^\varepsilon(t)|^a |\tilde{u}^\varepsilon(t)|^b \, d\theta = \int\limits_{E^\varepsilon} |u^\varepsilon(t)|^a |\tilde{u}^\varepsilon(t)|^b \, d\theta + o(1) \ ,$$

where

$$E^\varepsilon = \{\theta \in (-\pi, \pi) \ : \ |\theta - ct| < R^\varepsilon \varepsilon \ , \ |\theta - \tilde{c}t| < R^\varepsilon \varepsilon\} \ .$$

In view of the assumption on R^ε, this set is empty for ε small enough. This completes the proof. $\square$

6. The case of $\tilde{\mathcal{M}}(1)$

The manifold $\tilde{\mathcal{M}}(1)$ is a three-dimensional Kähler manifold, on which (5) admits three conservation laws in involution, which are Q, M, E. As we will see later, these conservation laws are generically independent on $\tilde{\mathcal{M}}(1)$, therefore the equation (S) is completely integrable on this manifold. We are going to solve this system explicitly, by introducing coordinates which are close to the action angle coordinates provided by the Liouville theorem (see Arnold [1]). Then we will establish some instability phenomena for large time.

6.1. **The evolution on $\tilde{\mathcal{M}}(1)$.** Let us make some preliminary calculations. Since the rank of H_u^2 is 2 if $u \in \tilde{\mathcal{M}}(1)$, the Cayley-Hamilton theorem reads

$$H_u^4 - \sigma_1 H_u^2 + \sigma_2 P_u = 0 \ . \tag{21}$$

Here, σ_1 is the trace of H_u^2 so it equals $Q + M$. Let us compute σ_2. Applying the above formula to the preimage $w \in \mathrm{Im}(H_u)$ of 1 introduced in subsection 4.3, we get

$$H_u^3(1) - (M + Q)u + \sigma_2 w = 0 \ . \tag{22}$$

Taking the scalar product of (22) with w, and using that $(u|w) = (H_u(1)|w) = (H_u(w)|1) = 1$, we infer

$$\sigma_2 = M\tilde{S} \ .$$

We now apply (21) to $1 \in \mathrm{Im}(H_u)$, and take the scalar product with 1. This yields

$$J_4 = (M + Q)Q - M\tilde{S}$$

or

$$E = Q^2 + 2M(Q - \tilde{S}) \ .$$

Consequently, we can use $M, Q, \tilde{S}$ rather than M, Q, E as our three conservation laws. For future reference, we introduce the solutions $r_\pm$ of the characteristic equation,

$$r^2 - \sigma_1 r + \sigma_2 = 0 \ ,$$

given by

$$r_\pm = \frac{1}{2}\left(Q + M \pm ((Q+M)^2 - 4M\tilde{S})^{1/2}\right) \ ,$$

and we set

$$\Omega = r_+ - r_- = ((M+Q)^2 - 4M\tilde{S})^{1/2} \ .$$

Proposition 7. *Let $u_0 \in \tilde{\mathcal{M}}(1)$, and let $u = \dfrac{az+b}{1-pz}$ be the corresponding solution of (5). One of the following two cases occurs :*

- *Either $Q = \tilde{S}$, and*

$$u_0(z) = a_0 \frac{z - \overline{p}}{1 - pz} \ , \ u(t,z) = \mathrm{e}^{-iQt} u_0(z) \ . \tag{23}$$

- *Or $Q > \tilde{S}$, and the evolution of a, b, p is given by*

$$i\dot{a} = Qa \ , \ i\dot{f}_\pm = r_\pm f_\pm \ , \tag{24}$$

  with

$$f_\pm := r_\pm b + M a \overline{p} \ .$$

  In particular, $|p|^2$ satisfies

$$|p|^2 = A + B\cos(\Omega t + \varphi)$$

  for some constants A, B, φ, and $|p|$ oscillates between the following values,

$$\rho_{\max} = \frac{M^{1/2} + \tilde{S}^{1/2}}{(M + Q + 2\sqrt{M\tilde{S}})^{1/2}} \ , \ \rho_{\min} = \frac{|M^{1/2} - \tilde{S}^{1/2}|}{(M + Q - 2\sqrt{M\tilde{S}})^{1/2}} \ . \tag{25}$$

Remark 2. *In the case (23), the solution u is called a stationary wave. We will classify such solutions in section 9.*

Proof. We already know that

$$i\dot{a} = Qa \ .$$

By corollary 3, we also know that $J_1 = b$ and J_3 satisfy

$$i\dot{J}_1 = J_3 \ , i\dot{J}_3 = J_5 \ ,$$

and $J_5 = \sigma_1 J_3 - \sigma_2 J_1$ in view of (21). Finally, J_3 is easily obtained by taking the scalar product of (22) with 1 and using Proposition 3,

$$J_3 = (M+Q)J_1 - M\tilde{S}(w|1) = (M+Q)b + Ma\overline{p} \ .$$

Setting

$$f_\pm := J_3 - r_\mp J_1 = r_\pm b + Ma\overline{p} \ ,$$

we finally obtain the system of linear ODE (24). Let us first investigate the particular case $r_+ = r_-$, which is equivalent to

$$(Q+M)^2 - 4M\tilde{S} = 0 \ .$$

Since $Q \geq \tilde{S}$ by the Cauchy-Schwarz inequality applied to u and w, we conclude that $r_+ = r_-$ is equivalent to

$$M = Q = \tilde{S} \ .$$

Using the Cauchy-Schwarz equality case, it is easy to check that $Q = \tilde{S}$ is equivalent to the collinearity of u_0 and w_0, namely

$$u_0(z) = a_0 \frac{z - \overline{p}}{1 - pz} \ .$$

Finally, from this expression of u_0, a simple computation gives $M = Q$, hence $r_+ = r_-$, and, since $|u_0|^2 = Q$ on $\mathbb{S}^1$, we get

$$u(t) = u_0 \mathrm{e}^{-iQt} \ .$$

In the case $r_+ \neq r_-$, we can recover $(a(t), b(t), p(t))$ from the variables $(a(t), f_\pm(t))$ and the conservation laws $(M, Q, \tilde{S})$. In particular,

$$M a \overline{p} = \frac{r_+ f_- \ - \ r_- f_+}{r_+ - r_-} \ . \tag{26}$$

Taking the modulus of both sides of (26), we conclude, in view of the differential equations satisfied by $f_\pm$, that

$$|p|^2 = A + B\cos(\Omega t + \varphi)$$

for some constants A, B, φ. Consequently, in view of (26), $|p|$ oscillates between the following values,

$$\rho_{\max} = \frac{r_+|f_-| + r_-|f_+|}{M\Omega \tilde{S}^{1/2}} \ , \ \rho_{\min} = \frac{|\, r_+|f_-| - r_-|f_+|\, |}{M\Omega \tilde{S}^{1/2}} \ .$$

Let us compute $|f_\pm|$ in terms of $M, Q, \tilde{S}$. Denote by (e_+, e_-) an orthonormal basis of $\mathrm{Im} H_u$ such that $\mathbb{C}e_\pm = \ker(H_u^2 - r_\pm)$. Up to multiplying $e_\pm$ by a suitable complex number of modulus 1, we may assume, using the $\mathbb{C}$-antilinearity of H_u, that

$$H_u e_\pm = \sqrt{r_\pm} e_\pm \ .$$

Then

$$1 = \overline{\zeta}_+ e_+ + \overline{\zeta}_- e_- \ , \ \zeta_\pm := (e_\pm|1) \ , \ u = H_u(1) = \sqrt{r}_+ \zeta_+ e_+ + \sqrt{r_-}\zeta_- e_- \ .$$

From

$$\begin{aligned} 1 &= |\zeta_+|^2 + |\zeta_-|^2 \ , \ Q = r_+|\zeta_+|^2 + r_-|\zeta_-|^2 \ , \\ J_1 &= \sqrt{r_+}\zeta_+^2 + \sqrt{r_-}\zeta_-^2 \ , \ J_3 = r_+\sqrt{r_+}\zeta_+^2 + r_-\sqrt{r_-}\zeta_-^2 \ , \end{aligned}$$

we obtain

$$|f_+| = \sqrt{r_+}(Q - r_-) \ , \ |f_-| = \sqrt{r_-}(r_+ - Q) \ ,$$

and finally (25), by a straightforward but tedious calculation. $\square$

In the next subsections, we shall take advantage of the oscillations of $|p|$ in establishing instability results.

6.2. **Large time estimates of H^s norms.** Our first instability result concerns large time behavior of H^s norms along trajectories of the cubic Szegö equation on $\tilde{\mathcal{M}}(1)$.

Corollary 5. *For every $u_0 \in \tilde{\mathcal{M}}(1)$, the solution u of (5) with $u(0) = u_0$ satisfies, for every $s > 1/2$,*

$$\limsup_{t\to\infty} \|u(t)\|_{H^s} < +\infty \ . \tag{27}$$

However, there exists a family $(u_0^\varepsilon)_{\varepsilon>0}$ of Cauchy data in $\tilde{M}(1)$, which converges in $\tilde{M}(1)$ for the $C^\infty(\mathbb{S}^1)$ topology as $\varepsilon \to 0$, and $K > 0$ such that the corresponding solutions u^ε satisfy

$$\forall \varepsilon > 0, \exists t^\varepsilon > 0 : t^\varepsilon \to \infty \ , \ \forall s > \frac{1}{2} \ , \|u^\varepsilon(t^\varepsilon)\|_{H^s} \geq K(t^\varepsilon)^{2s-1} \ . \tag{28}$$

Proof. Writing as before

$$u(t) = \frac{a(t)z + b(t)}{1 - p(t)z} \ ,$$

we already know that $a(t)$ and $b(t)$ are bounded because of the conservation of $Q(u(t))$, so the blow up of the H^s norm for large $|t|$ would only come from the fact that $|p(t)|$ approaches 1. But this cannot happen since, by formula (25),

$$\max_t |p(t)| = \rho_{\max} < 1$$

if $Q > \tilde{S}$. The other case $Q = \tilde{S}$ corresponds to (23), for which $p(t) = p(0)$.

Let us turn to the second assertion. Consider the family of Cauchy data $\{u_0^\varepsilon\}_{0<\varepsilon<1}$ given by

$$u_0^\varepsilon(z) = z + \varepsilon$$

and let us look at the regime $\varepsilon \to 0$. Then a simple computation from the previous formulae shows that

$$|p(t)|^2 = \frac{2}{4+\varepsilon^2}(1 - \cos(\varepsilon t\sqrt{4+\varepsilon^2})).$$

On the other hand, using Fourier expansion, we have, as $|p|$ approaches 1,

$$\|u\|_{H^s}^2 \simeq \frac{|a + bp|^2}{(1-|p|^2)^{2s+1}} = M\frac{1}{(1-|p|^2)^{2s-1}}$$

since

$$M(u) = \frac{|bp + a|^2}{(1-|p|^2)^2} \ .$$

In our particular case, $M(u) = 1$ and we get, for $t^\varepsilon = \dfrac{\pi}{\varepsilon\sqrt{4+\varepsilon^2}}$,

$$\|u(t^\varepsilon)\|_{H^s}^2 \simeq \frac{1}{(1-|p(t^\varepsilon)|^2)^{2s-1}} \simeq C(t^\varepsilon)^{2(2s-1)}.$$

This completes the proof. $\square$

Remark 3. *Property (28) can be seen as a quantitative version of an instability property proved in* [11] *for NLS on the two dimensional torus: bounded data in C^∞ may yield large solutions in H^s for large time. However, as shown by (27), this may happen even if the H^s norms stay bounded on each individual trajectory, and moreover in the case of a completely integrable system. Notice that this phenomenon can occur with arbitrarily small data, since multiplying the Cauchy data by a parameter δ amounts to replace the solution $u(t)$ of (5) by $\delta u(\delta^2 t)$.*

6.3. **Orbital instability of stationary waves.** Our next instability result concerns the stationary waves in $\tilde{\mathcal{M}}(1)$.

Corollary 6. *For each stationary wave u_0 of $\tilde{\mathcal{M}}(1)$, there exists a sequence u_0^ε which converges to u_0 in C^∞ such that, for every $r \in (0,1)$, there exists t^ε such that the limit points in $H^{1/2}_+$ of $u^\varepsilon(t^\varepsilon)$ are of the form*

$$v = \alpha\,\frac{z-\overline{q}}{1-qz}\ ,\ \ |\alpha| = \|u_0\|_{L^2}\ ,\ \ |q| = r.$$

Proof. First recall that if $v = \dfrac{az+b}{1-pz}$ then the conservation laws are given by $M = \dfrac{|bp+a|^2}{(1-|p|^2)^2}$, $Q = \dfrac{|bp+a|^2}{(1-|p|^2)} + |b|^2$ and $\tilde{S} = |a|^2$.

Let $u_0 = a\,\dfrac{z-\overline{p}}{1-pz}$ be a stationary wave of $\tilde{\mathcal{M}}(1)$. Define, for $0 < \varepsilon < 1$,

$$u_0^\varepsilon = a\,\frac{(1-\varepsilon)z - \overline{p}(1-\varepsilon/2)}{1-p(1+\epsilon/2)z}\ .$$

It is clear that such a sequence converges to u_0 in C^∞. By Proposition 7, for any fixed ε, the corresponding solution u^ε may be written as $\dfrac{a^\varepsilon z + b^\varepsilon}{1-p^\varepsilon z}$ where $|p^\varepsilon|$ oscillates between $\rho^\varepsilon_{\min}$ and $\rho^\varepsilon_{\max}$ given by (25). Computing these two bounds in terms of ε, it is easy to show that $\rho^\varepsilon_{\max}$

tends to 1 and $\rho^\varepsilon_{\min}$ tends to 0 as ε goes to 0. Precisely, we have,

$$\begin{aligned}
M &= |a|^2\frac{|(1-\varepsilon)-|p|^2(1-\varepsilon^2/4)|^2}{(1-|p|^2(1+\varepsilon/2)^2)^2}\\
&= |a|^2\,(1-2\varepsilon)+\mathcal{O}(\varepsilon^2)\\
Q &= |a|^2\left(\frac{|(1-\varepsilon)-|p|^2(1-\varepsilon^2/4)|^2}{(1-|p|^2(1+\varepsilon/2)^2)}+|p|^2(1-\varepsilon/2)^2\right)\\
&= |a|^2(1-2\varepsilon)+\mathcal{O}(\varepsilon^2)\\
\tilde S &= |a|^2(1-\varepsilon)^2=|a|^2(1-2\varepsilon)+\mathcal{O}(\varepsilon^2).
\end{aligned}$$

¿From these estimates, we get $\Omega=\mathcal{O}(\varepsilon)$, $\sqrt{M}-\sqrt{\tilde S}=\mathcal{O}(\varepsilon^2)$ and

$$\rho^\varepsilon_{\max}=1+\mathcal{O}(\varepsilon^2)\ ,\ \rho^\varepsilon_{\min}=\mathcal{O}(\varepsilon)\ .$$

In particular, for every $r\in(0,1)$, one can choose t^ε such that $|p^\varepsilon(t^\varepsilon)|=r$. As the $H^{1/2}$-norms of $u^\varepsilon(t)$ are bounded, $u^\varepsilon(t^\varepsilon)$ has limit points in the weak $H^{1/2}$-topology. Let v_∞ be such a limit point. Since $p^\varepsilon(t^\varepsilon)$ stays on the circle of radius r, the convergence is strong and v_∞ belongs to $\tilde{\mathcal{M}}(1)$. Moreover, $Q(v_\infty)=\tilde S(v_\infty)$, hence v_∞ is given by (23). This completes the proof.

□

We will pursue our study of large time behavior in section 7.

7. Large time behavior on $\mathcal{M}(N)$

By Corollary 5, every solution on $\tilde{\mathcal{M}}(1)$ satisfies

$$\sup_{t\in\mathbb{R}}\|u(t)\|_{H^s}<+\infty$$

for $s\geq 0$. We prove that it is a generic situation on $\mathcal{M}(N)$. A similar statement holds on $\tilde{\mathcal{M}}(N-1)$.

Theorem 7.1. *For every integer N, define*

$$V_N=\{u_0\in\mathcal{M}(N);\ \det(J_{2(m+n)}(u_0))_{1\leq m,n\leq N}=0\}\ .$$

Then V_N is a proper real analytic subvariety of $\mathcal{M}(N)$ and, for every $u_0\in\mathcal{M}(N)\setminus V_N$, for every $s\geq 0$,

$$\sup_{t\in\mathbb{R}}\|u(t)\|_{H^s}<+\infty. \tag{29}$$

In particular, (29) holds for every u_0 outside a closed subset of measure 0. A similar statement holds on $\tilde{\mathcal{M}}(N-1)$, with

$$\tilde V_{N-1}:=V_N\cap\tilde{\mathcal{M}}(N-1).$$

Proof. For every $u\in H^{1/2}_+$, we consider the polynomial expression

$$F_N(u)=\det(J_{2(m+n)}(u))_{1\leq m,n\leq N}\ .$$

Notice that $F_N(u)=0$ if and only if the vectors $H_u^{2k}(1)\ ,\ k=1,\cdots,N$ are linearly dependent. In particular, F_N is identically 0 on $\mathcal{M}(J)$ for

$J < N$. On the other hand, we shall see that F_N is not identically 0 on $\mathcal{M}(N)$. In fact, one can prove the following slightly stronger result, which we state as a lemma for further references.

Lemma 4. *The vectors $H_u^{2k}(1), k = 1, \cdots, N$, are generically independent on $\tilde{\mathcal{M}}(N-1)$ and on $\mathcal{M}(N)$.*

Proof. Indeed, if

$$u(z) = z^{N-1} + z^{N-2} ,$$

$u \in \tilde{\mathcal{M}}(N-1)$ and a simple computation shows that the matrix of the system $1, H_u^2(1), H_u^4(1), \cdots, H_u^{2(N-1)}(1)$, in the basis $(z^j)_{0\le j\le N-1}$ is triangular, hence these vectors are independent. Applying H_u^2, which is one to one on $\mathrm{Im}(H_u)$, the vectors $H_u^{2k}(1), k = 1, \cdots, N$, are independent as well, and $F_N(u) \neq 0$. Since $\tilde{\mathcal{M}}(N-1)$ and $\mathcal{M}(N)$ are connected, this completes the proof. □

Theorem 7.1 is then a consequence of the following Lemma.

Lemma 5. *If $u_0 \in \mathcal{M}(N) \setminus V_N$, the level set*

$$L_N(u_0) := \{u \in \mathcal{M}(N) : J_{2n}(u) = J_{2n}(u_0), 1 \le n \le 2N\}$$

is a compact subset of $\mathcal{M}(N)$.
If $u_0 \in \tilde{\mathcal{M}}(N-1) \setminus V_N$, the level set

$$\tilde{L}_{N-1}(u_0) := \{u \in \tilde{\mathcal{M}}(N-1) : J_{2n}(u) = J_{2n}(u_0), 1 \le n \le 2N-1\}$$

is a compact subset of $\tilde{\mathcal{M}}(N-1)$.

Proof. We just prove the statement for $\mathcal{M}(N)$. Let $u_0 \in \mathcal{M}(N) \setminus V_N$ and $u \in L_N(u_0)$. Let us first prove that $M(u) = M(u_0)$. By the Cayley-Hamilton theorem applied to H_u^2 on $\mathrm{Im}(H_u)$,

$$H_u^{2N} = \sum_{j=1}^{N} (-1)^{j-1} \sigma_j(u) H_u^{2(N-j)} .$$

Applying this identity to $H_u^{2p}(1)$ for $p = 1, \cdots, N$ and taking the scalar product with 1, we obtain a system of N linear equations in the σ_j's,

$$J_{2(N+p)}(u) = \sum_{j=1}^{N} (-1)^{j-1} \sigma_j(u) J_{2(N+p-j)} \ , \ 1 \le p \le N.$$

The determinant of this system is $\det(J_{2(m+n)}(u))_{0\le m\le N-1, 1\le n\le N}$, which, by the above identity, is $(-1)^{N-1}F_N(u)/\sigma_N(u)$, hence is not zero — notice that $\sigma_N(u) \neq 0$, since H_u^2 is one to one on $\mathrm{Im}(H_u)$. Solving this system, we conclude that each $\sigma_j(u)$ is a universal function of $(J_{2n}(u))_{1\le n\le 2N}$. Since $\sigma_1 = M + J_2$, this proves the claim. We infer that every sequence of $L_N(u_0)$ is bounded in $H^{1/2}$, hence has limit points for the weak topology of $H^{1/2}$. Let v be such a limit point. As a limit point of a sequence of $\mathcal{M}(N)$, v belongs to $\cup_{J\le N}\mathcal{M}(J)$.

On the other hand, since each J_{2n} is continuous for the weak topology of $H^{1/2}$, $J_{2n}(v) = J_{2n}(u_0)$ for $n = 1, \cdots, 2N$. In particular, $F_N(v) = F_N(u_0) \neq 0$, whence $v \in \mathcal{M}(N)$ and finally $v \in L_N(u_0)$. $\square$

The proof of Theorem 7.1 is completed by observing that the flow of (5) conserves the level sets L_N, and that the zeroes of the denominator of elements of a compact subset of $\mathcal{M}(N)$ do not approach the unit circle. $\square$

Corollary 7. *For every $u_0 \in \mathcal{M}(2)$, $s \geq 0$, (29) holds.*

Proof. In view of Theorem 7.1, it is enough to consider the case $F_2(u) = 0$, which is equivalent to the collinearity of $H_u^2(1)$ and of $P_u(1)$,

$$H_u^2(1) = \frac{Q}{1-S} P_u(1) \ .$$

If $P_u(1) = 1 \in \mathrm{Im}(H_u)$, then $|u|^2 = Q$ and u is a stationary wave by Proposition 8. If $P_u(1) \neq 1$, by Proposition 2, the function $v = 1 - P_u(1)$ satisfies

$$i\partial_t v = |u|^2 v = \frac{Q(1+S)}{1-S} v - \frac{Q}{1-S} v^2 - \frac{QS}{1-S} \ .$$

Notice that $S = (v|1)$ is a particular solution of this Riccati equation. Hence we can solve it explicitly and observe that v is a periodic function of t with period $2\pi/Q$. Since, by Proposition 1,

$$v(t,z) = p_1(t)p_2(t) \frac{(z - \overline{p_1}(t))(z - \overline{p_2}(t))}{(1 - p_1(t)z)(1 - p_2(t)z)} \ ,$$

we conclude that p_1, p_2 are periodic as well, hence cannot approach the unit circle. $\square$

8. The Szegö hierarchy

In this section, we show that the conservation laws J_{2n} satisfy the Poisson commutation relations

$$\{J_{2n}, J_{2p}\} = 0 \ ,$$

and that J_{2n} defines a global Hamiltonian flow for every n. In fact, we prove that, for every n, there exists a skew symmetric operator $B_{u,n}$ such that the pair $(H_u, B_{u,n})$ is a Lax pair for this Hamiltonian flow. The last part of the section is devoted to proving that functions $(J_{2n})_{1 \leq n \leq 2N}$ are generically independent on $\mathcal{M}(N)$, and that functions $(J_{2n})_{1 \leq n \leq 2N+1}$ are generically independent on $\tilde{\mathcal{M}}(N)$. This will complete the proof of Theorem 1.2 in the introduction.

Theorem 8.1. *Let $s > \frac{1}{2}$. The map $u \mapsto J_{2n}(u)$ is smooth on H^s_+ and its Hamiltonian vector field is given by*

$$X_{J_{2n}}(u) = \frac{1}{2i} \sum_{j=0}^{n-1} H_u^{2j}(1) H_u^{2n-2j-1}(1) \ . \tag{30}$$

Moreover,

$$H_{iX_{J_{2n}}(u)} = H_u A_{u,n} + A_{u,n} H_u$$

where $A_{u,n}$ *is the self adjoint operator*

$$A_{u,n}(h) = \frac{1}{4}\left(\sum_{j=0}^{2n-2} H_u^j(1)\Pi(\overline{H_u^{2n-2-j}(1)}h) - \sum_{k=1}^{n-1}(h|H_u^{2k-1}(1))H_u^{2n-2k-1}(1)\right) .$$

Proof. Introduce, for x real and $|x|$ small enough, the generating functions,

$$w(x) = (1 - xH_u^2)^{-1}(1) = \sum_{n=0}^{\infty} x^n H_u^{2n}(1)$$

and

$$J(x,u) = (w(x)|1) = \sum_{n=0}^{\infty} x^n J_{2n}(u) .$$

We have

$$\begin{aligned} d_u J(x,u).h &= ((1 - xH_u^2)^{-1}x(H_uH_h + H_hH_u)(1 - xH_u^2)^{-1}(1)|1) \\ &= x[(H_uH_hw(x)|w(x)) + (H_hH_uw(x)|w(x))] = 2x\mathrm{Re}(h|w(x)H_uw(x)) \\ &= \omega(h|X(x)) \end{aligned}$$

with

$$X(x) = \frac{x}{2i} w(x)H_uw(x) .$$

Identifying the coefficients of x^n, we get formula (30). The second part of the proof relies on the following lemma.

Lemma 6. *We have the following identity,*

$$H_{aH_u(a)}(h) = H_u(a)H_a(h) + H_u(a\Pi(\overline{a}h) - (h|a)a) .$$

Proof.

$$H_{aH_u(a)}(h) = \Pi(aH_u(a)\overline{h}) = H_u(a)H_a(h) + \Pi(H_u(a)(1-\Pi)(a\overline{h})) .$$

On the other hand,

$$(1-\Pi)(a\overline{h}) = \overline{\Pi(\overline{a}h)} - (a|h) .$$

The lemma follows by plugging the latter formula into the former one. ∎

Let us complete the proof. Using the identity

$$w(x) = 1 + xH_u^2 w(x),$$

and Lemma 6 with $a = H_u(w)$, we get

$$\begin{aligned} H_{wH_u(w)}(h) &= H_{H_u(w)}(h) + xH_{H_u(w)H_u^2(w)}(h) \\ &= H_{H_u(w)}(h) + xH_u^2(w)H_{H_u(w)}(h) + \\ &+ xH_u\left(H_u(w)\Pi(\overline{H_u(w)}h) - (h|H_u(w))H_u(w)\right) \\ &= wH_{H_u(w)}(h) + xH_u\left(H_u(w)\Pi(\overline{H_u(w)}h) - (h|H_u(w))H_u(w)\right) \\ &= w\Pi(\overline{w}H_uh) + xH_u\left(H_u(w)\Pi(\overline{H_u(w)}h) - (h|H_u(w))H_u(w)\right) . \end{aligned}$$

We therefore have obtained

$$H_{wH_u(w)} = G_uH_u + H_uD_u$$

where G_u and D_u are the following self adjoint operators,

$$G_u(h) = w\Pi(\overline{w}h) \ , \ D_u(h) = x\left(H_u(w)\Pi(\overline{H_u(w)}h) - (h|H_u(w))H_u(w)\right) .$$

Consequently, since $H_{wH_u(w)}$ is self adjoint,

$$H_{wH_u(w)} = C_uH_u + H_uC_u$$

with

$$C_u = \frac{1}{2}(G_u + D_u) .$$

Identifying the coefficients of x^n in

$$H_{iX(x)} = \frac{x}{2}H_{w(x)H_uw(x)} ,$$

we infer the desired formula for $A_{u,n}$.

$\square$

Corollary 8. *Let $s > 1$. For every $u_0 \in H^s_+$, there exists a unique solution $u \in C(\mathbb{R}, H^s_+)$ of the Cauchy problem*

$$\partial_t u = X_{J_{2n}}(u) \ , \ u(0) = u_0 . \tag{31}$$

Moreover, u satisfies

$$\partial_t H_u = [B_{u,n}, H_u] , \tag{32}$$

with

$$B_{u,n}(h) = \frac{-i}{4}\left(\sum_{j=0}^{2n-2} H_u^j(1)\Pi(\overline{H_u^{2n-2-j}(1)}h) - \sum_{k=1}^{n-1}(h|H_u^{2k-1}(1))H_u^{2n-2k-1}(1)\right) .$$

Finally, we have the commutation identity

$$\{J_{2n}, J_{2p}\} = 0 . \tag{33}$$

Proof. The local-in-time solvability of the Cauchy problem is an easy consequence of the fact that H^s is an algebra. Moreover, to prove global existence, it is enough to establish that the L^∞ norm of u does not blow up in finite time. In view of Theorem 8.1, u satisfies equation (32) on its interval of existence. Since $B_{u,n}$ is skew symmetric, this implies that

$Tr(|H_u|)$ is conserved, and consequently, by Peller's theorem [24], that the norm of u in $B^1_{1,1}$ is bounded, and so is the L^∞ norm, whence the global existence, by an elementary Gronwall argument.

It remains to prove the commutation identity (33). This is equivalent to the fact that J_{2p} is a conservation law of the Hamiltonian flow of J_{2n}. The latter fact is a consequence, as in section 3, of equation (32), and of the formula

$$B_{u,n}(1) = \frac{-i}{4}\sum_{\ell=0}^{n-1} J_{2n-2\ell-2} H_u^{2\ell}(1).$$

□

We conclude this section with a complete integrability result.

Corollary 9. *Let $N \geq 1$. The following properties hold.*

1. *The functions $J_{2k}\,, k = 1, \cdots, 2N$ are independent in the complement of a closed subset of measure 0 of $\mathcal{M}(N)$.*
2. *The functions $J_{2k}\,, k = 1, \cdots, 2N+1$ are independent in the complement of a closed subset of measure 0 of $\tilde{\mathcal{M}}(N)$.*

Consequently, for generic Cauchy data in $\mathcal{M}(N)$ and in $\tilde{\mathcal{M}}(N)$, the solution of (5) is quasiperiodic.

Proof. First notice that $X_{J_{2n}}$ is tangent to $\mathcal{M}(N)$ and to $\tilde{\mathcal{M}}(N)$. This can be seen either from the explicit expression (30) of $X_{J_{2n}}$ compared to the explicit description of the tangent spaces of $\mathcal{M}(N)$ and $\tilde{\mathcal{M}}(N)$ in section 4, or as a consequence of the Kronecker theorem,

$$\mathcal{M}(N) = \{u : rk(H_u) = N\}\,,\ \tilde{\mathcal{M}}(N) = \{u \in \mathcal{M}(N{+}1) : 1 \in \mathrm{Im}(H_u)\}\,,$$

compared with the Lax pair property for the flow of $X_{J_{2n}}$ proved in Corollary 8. Consequently, the functions J_{2k} restricted to the symplectic manifolds $\mathcal{M}(N)$ and to $\tilde{\mathcal{M}}(N)$ are in involution. Therefore the second statement of the corollary is reduced to properties (1) and (2). Notice that property (1) holds for $N = 1$. Indeed, the linear dependence of J_2 and J_4 at u is equivalent to the fact that u is a stationary wave, which, on $\mathcal{M}(1)$, means that u is a constant. We shall prove that, for all N, property (1) implies property (2) and that property (2) implies property (1) for $N+1$. This will complete the proof by induction.

We first prove that property (2) for N implies property (1) for $N+1$. We represent the current generic point $u \in \mathcal{M}(N+1)$ as

$$u(z) = \frac{A(z)}{B(z)}\ ,\ A \in \mathbb{C}_N[z],\ d(A) = N\ ,$$

with $B(z) = bz^{N+1} + \tilde{B}(z)$, $\tilde{B} \in \mathbb{C}_N[z]$. In this representation, $\tilde{\mathcal{M}}(N)$ is characterized by the cancellation of the holomorphic coordinate b.

Notice that $S = |b|^2$. Fix $u_0 \in \tilde{\mathcal{M}}(N)$ such that the differential form

$$\alpha := \bigwedge_{k=1}^{2N+1} dJ_{2k}$$

satisfies $\alpha(u_0) \neq 0$ on $T_{u_0}\tilde{\mathcal{M}}(N) = \ker db(u_0)$. In a small neighborhood U of u_0 in $\mathcal{M}(N+1)$, define 2N+1 vector fields $Y_k, k = 1, \cdots, 2N+1$, such that, for every $u \in U$, $(Y_k(u))_{1\le k\le 2N+1}$ is a basis of $\ker(db(u))$. Since

$$\alpha(u_0)(Y_1(u_0), \cdots, Y_{2N+1}(u_0)) \neq 0,$$

this is still true near u_0. On the other hand, since $S = |b|^2$, $dS.Y_j = 0$ by construction. Hence

$$\begin{aligned}(dS \wedge \alpha)\left(b\frac{\partial}{\partial b}, Y_1, \cdots, Y_{2N+1}\right) &= dS\left(b\frac{\partial}{\partial b}\right)\alpha(Y_1, \cdots, Y_{2N+1}) \\ &= 2S\,\alpha(Y_1, \cdots, Y_{2N+1})\ ,\end{aligned}$$

which does not cancel on $U \setminus \tilde{\mathcal{M}}(N)$. This shows that the functions $S, J_{2k}, k = 1, \cdots, 2N+1$ are generically independent on $\mathcal{M}(N+1)$. In view of Lemma 4, we also know that the $N+1$ vectors $H_u^{2k}(1), k = 1, \cdots, N+1$ are generically linearly independent. Since H_u is one to one on $\mathrm{Im}(H_u)$, this is true as well for the vectors $H_u^{2k+1}(1), k = 0, \cdots, N$, in other words

$$\det(J_{2(m+n+1)})_{0\le m,n\le N} \neq 0$$

generically on $\mathcal{M}(N+1)$. Now apply the Cayley-Hamilton Theorem to H_u^2, as we did for the proof of Lemma 5. For every $p = 1, \cdots, N+1,$, we obtain

$$J_{2(N+1+p)} = \sum_{j=1}^{N+1}(-1)^{j-1}\sigma_j J_{2(N+1-j+p)}\ . \tag{34}$$

Solving this linear system, we infer that, locally at generic points,

$$\sigma_j = F_j(J_{2k}, k = 1, \cdots, 2N+2)$$

where F_j is real analytic. Applying again (34) for $p = 0$, and observing that $J_0 = 1 - S$ and $\sigma_N \neq 0$ since H_u is one to one on $\mathrm{Im}(H_u)$, we obtain, locally at generic points,

$$S = G(J_{2k}, k = 1, \cdots, 2N+2)$$

where G is real analytic. This implies that the functions $J_{2k}, k = 1, \cdots, 2N+2$, are generically independent on $\mathcal{M}(N+1)$, which is property (2) for $N+1$.

The proof that property (1) implies property (2) is quite similar, so we just sketch it. First we enlarge $\tilde{\mathcal{M}}(N)$ as a connected holomorphic manifold of the same dimension, which contains a dense open subset of $\mathcal{M}(N)$ as a hypersurface. This can be realized by considering the

manifold $\tilde{\mathcal{M}}'(N) = \tilde{\mathcal{M}}(N) \cup \mathcal{M}(N) \setminus \tilde{\mathcal{M}}(N-1)$ which consists of rational functions u of the form

$$u(z) = \frac{A(z)}{B(z)} ,$$

with $A \in \mathbb{C}_N[z]$, $B \in \mathbb{C}_N[z]$, $B(0) = 1$, $d(A) = N$ or $d(B) = N$, A and B have no common factors, and $B(z) \neq 0$ if $|z| \leq 1$. The coefficient a of z^N in the numerator A defines a holomorphic coordinate on $\tilde{\mathcal{M}}'(N)$, and $\mathcal{M}(N)$ is defined by the equation $a = 0$. Moreover, $\tilde{S} = |a|^2$ is a conservation law. Starting from a generic point $u_0 \in \mathcal{M}(N)$, we prove similarly that the functions $\tilde{S}, J_{2k}, k = 1, \cdots 2N$, are generically independent on $\tilde{\mathcal{M}}'(N)$. Then we infer the generic independence of $J_{2k}, k = 1, \cdots 2N+1$, by using again the Cayley-Hamilton theorem for H_u^2.

It is now easy to conclude, generically on the data in $\mathcal{M}(N)$ or $\tilde{\mathcal{M}}(N)$, that the solution of equation (5) is quasiperiodic. Let us sketch the argument for $\mathcal{M}(N)$, for instance. By Lemma 5, for generic u_0 in $\mathcal{M}(N)$, the level set

$$L_N(u_0) := \{u \in \mathcal{M}(N) : J_{2n}(u) = J_{2n}(u_0), 1 \leq n \leq 2N\}$$

is compact. Moreover, from the generic independence of the functions J_{2n} combined with the Sard theorem, for generic $u_0 \in \mathcal{M}(N)$, the vector $(J_{2n}(u_0))_{1 \leq n \leq 2N}$ is a regular value of the mapping

$$u \mapsto (J_{2n}(u))_{1 \leq n \leq 2N} .$$

We conclude from standard arguments — see for instance [1], that, generically on $u_0 \in \mathcal{M}(N)$, the level set $L(u_0)$ is a finite union of $2N$ dimensional Lagrangian tori, on which the evolution defined by (5) is quasiperiodic. □

9. Traveling waves

We start with some basic definitions. General definitions can be found in [14], for example.

Definition 3. *A solution u of (5) is said to be a traveling wave if there exists $\omega, c \in \mathbb{R}$ such that*

$$u(t, z) = e^{-i\omega t} u(0, e^{-ict} z)$$

for every $t \in \mathbb{R}$. We shall call ω the pulsation of u, and c the velocity of u.

Equivalently, u is a traveling wave with pulsation ω and angular velocity c if and only if it satisfies at time $t = 0$ — hence at every time— the following equation,

$$cDu + \omega u = \Pi(|u|^2 u) . \tag{35}$$

In the sequel, a solution $u \in H^{1/2}_+$ of equation (35) will be called as well a traveling wave of pulsation ω and of velocity c. Notice that equation (35) is variational : traveling waves of pulsation ω and velocity c are the critical points of the functional

$$u \in H^{1/2}_+ \mapsto cM(u) + \omega Q(u) - \frac{1}{2}E(u) \ .$$

For example, from Proposition 5, we know that elements of $\mathcal{M}(1)$ are characterized as minimizers of

$$u \in H^{1/2}_+ \mapsto Q(u)^2 + 2M(u)Q(u) - E(u) \ ,$$

so that we recover that they are traveling waves with

$$\omega = Q(u) + M(u) \ , \ c = Q(u) \ .$$

9.1. **Characterization of stationary waves.** Stationary waves are traveling waves with velocity c equal to 0. They are particularly easy to characterize.

Proposition 8. *Let $u_0 \in H^{\frac{1}{2}}_+ \setminus \{0\}$. Then $u(t) = e^{-i\omega t}u_0$ solves (S) if and only if*

$$|u_0|^2 = \omega \text{ a.e.} \quad \text{on } \mathbb{S}^1 \ ,$$

or equivalently

$$u_0(z) = \alpha \prod_{j=1}^{N} \frac{z - \overline{p}_j}{1 - p_j z}$$

for some $p_1, \dots, p_N$ in the unit disc, and α is a complex number such that $|\alpha|^2 = \omega$.

Proof. Indeed, $\Pi(|u_0|^2 u_0) = \omega u_0$ means

$$|u_0|^2 u_0 - \omega u_0 \ \perp \ L^2_+$$

which implies $|u_0|^4 - \omega |u_0|^2 = 0$, or $|u_0|^2 = \omega$. In other words, $\varphi := \omega^{-1/2} u_0$ is an inner function in the sense of Beurling. Since $\varphi \in H^{1/2}_+$, the finiteness of

$$(D\varphi|\varphi) = \int_{\mathbb{S}^1} \frac{\varphi'(z)}{\varphi(z)} \frac{dz}{2i\pi}$$

implies, by Rouché's theorem, that φ has only a finite number of zeroes in the unit disc, therefore is a finite Blaschke product, as claimed. □

As it is well known (see *e.g.* [25], Chapter 17), any inner function may be written as a product of a Blaschke product and of

$$\exp\left(- \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t) \right)$$

where μ is a singular measure with respect to the Lebesgue measure. The simplest cases are

$$u_0(z) = \prod_{j=1}^{\infty} \frac{z - \overline{p_j}}{1 - p_j z}\ ,\ |p_j| < 1,\ \sum_{j=1}^{\infty}(1 - |p_j|) < \infty,$$

$$u_0(z) = \exp\left(-\frac{1+z}{1-z}\right) \qquad .$$

Let us emphasize that these particular solutions do not belong to $H^{1/2}_+$. Hence, these examples show that there exists a larger family of non smooth solutions of (S), which does not fit with the existence result of Theorem 2.1 and therefore calls for the construction of a flow map on a wider phase space. In view of the BMO conservation law derived from the Lax pair and Nehari's Theorem, a natural candidate for this phase space is BMO_+. This is a very interesting open question.

9.2. **Characterization of traveling waves.** We now focus on the case of a non zero velocity. The main result of this section is the following.

Theorem 9.1. *A function $u \in H^{1/2}_+$ is a traveling wave with a velocity $c \in \mathbb{R}^*$ and with a pulsation $\omega \in \mathbb{R}$ if and only if there exist non negative integers N, $\ell \in \{0, 1, \dots, N-1\}$, and complex numbers $p \in \mathbb{C}$ with $0 < |p| < 1$ and $\alpha \in \mathbb{C}$, such that*

$$u(z) = \frac{\alpha z^\ell}{1 - p^N z^N}.$$

Proof. We first reformulate the soliton equation (35) in terms of the Hankel operator H_u. Introducing the operator

$$A = D - \frac{1}{c} T_{|u|^2}$$

we observe from (13) that (35) is equivalent to

$$AH_u + H_u A + \frac{\omega}{c} H_u + \frac{1}{c} H_u^3 = 0\ . \tag{36}$$

The operator $\tilde{A} = A + \frac{1}{2c} H_u^2$ is selfadjoint on L^2_+, bounded from below and with a compact resolvent. Therefore it admits an orthonormal basis of eigenfunctions associated to a sequence of real eigenvalues tending to $+\infty$. Since (36) is equivalent to

$$\tilde{A} H_u + H_u \tilde{A} = -\frac{\omega}{c} H_u\ ,$$

we observe that

$$\tilde{A}\varphi = \lambda \varphi$$

yields

$$\tilde{A} H_u \varphi = -(\frac{\omega}{c} + \lambda) H_u \varphi$$

and the boundedness of $\tilde{A}$ from below implies $H_u\varphi = 0$ for λ large enough. Consequently, H_u has finite rank, and therefore u is a rational function by the Kronecker theorem (see appendix 3 for an elementary proof). The main step is now to prove the following result.

Proposition 9. *There exists $\lambda \in \mathbb{R}$ so that $H_u^2(u) = \lambda u$.*

Assume this proposition is proved, and let us show how to complete the proof of Theorem 9.1. We may assume that $1 \notin \mathrm{Im} H_u$, otherwise Proposition 9 would lead to $H_u^2(1) = \lambda$ which implies that $|u|^2 = \lambda$ and hence that u is a stationary wave. Denote by N the rank of H_u. Notice that (36) implies

$$[A, H_u^2] = 0 \tag{37}$$

therefore the range of H_u^2 — which is also the range of H_u— is invariant through the action of A.

As $1 \notin \mathrm{Im} H_u$, Proposition 9 reads $H_u^2(1) = \lambda P_u(1)$ with $\lambda = \frac{Q}{1-S}$. Setting $v = 1 - P_u(1)$ as in subsection 4.2, we have

$$|u|^2 = H_u^2(1) + \overline{H_u^2(1)} - Q = \frac{Q}{1-S}(2 - v - \overline{v}) - Q \ .$$

On the other hand, as v belongs to the kernel of H_u, we have from (36),

$$H_u A(v) = 0 \ .$$

But

$$A(v) = -\frac{1}{c} H_u^2(1) - A P_u(1) \ \in \mathrm{Im}(H_u).$$

We conclude that $A(v) = 0$, which reads, since $\overline{u}v$ is holomorphic from Lemma 1,

$$Dv = \frac{1}{c}|u|^2 v \ .$$

¿From Rouché 's theorem, we infer

$$Q = Nc.$$

Eventually, we get

$$Dv = \frac{N}{1-S}(2 - v - \overline{v})v - Nv = \frac{N(1+S)}{1-S}v - \frac{N}{1-S}v^2 - \frac{NS}{1-S}$$

since $|v|^2 = S$. Notice that the constant S is a particular solution of this Riccati equation. Solving this equation, we get, for some constant B,

$$v = 1 - \frac{(1-S)B}{B + z^N} \ .$$

¿From this formula, we have

$$H_u^2(1) = \frac{Q}{1-S}(1 - v) = \frac{Q}{1 - p^N z^N}$$

for some constant p which is necessarily of modulus less than 1 since $H_u^2(1)$ is holomorphic in the unit disc. It remains to use that u is solution to the equation

$$cDu + \omega u = H_u^3(1) + uH_u^2(1) - Qu$$

to get that

$$cDu + \omega u = u\left(\frac{QS}{1-S} + \frac{Q}{1-p^N z^N}\right).$$

This is an ordinary first order differential equation, which can be rewritten as

$$D\log(u) = D\log(1-p^N z^N)^{-1} + N\left(\frac{1}{1-S} - \frac{\omega}{Q}\right) D\log z\ .$$

By Rouché's theorem, since u is a rational function with no poles in the unit disc and at most $N-1$ zeroes, we have

$$N\left(\frac{1}{1-S} - \frac{\omega}{Q}\right) = \ell \in \{0, 1, \cdots, N-1\}\ .$$

Coming back to the equation on u, this proves the claim.

9.2.1. *Proof of Proposition 9.* We now turn to the main step of the proof. Because of (37), there exists an orthonormal basis of $\mathrm{Im}(H_u)$ which consists of common eigenvectors to A and to H_u^2. Our strategy is to describe precisely the corresponding joint spectrum. Let us introduce some notation. For $\gamma > 0$, set

$$E_{\lambda,\gamma} = \ker(A-\lambda) \cap \ker(H_u^2 - \gamma),$$

and define

$$\Sigma = \{(\lambda, \gamma) \in \mathbb{R} \times \mathbb{R}_+^* \ :\ E_{\lambda,\gamma} \neq \{0\}\ \}\ .$$

The following two lemmas give important information about Σ. The first one takes advantage of the relationship with the shift.

Lemma 7. (1) *Assume $A\varphi = \lambda\varphi$.*
If $(\varphi|1) = 0$, then $\varphi = z\psi$ with $A\psi = (\lambda-1)\psi$.
If $(z\varphi|H_u^2(1)) = 0$, then $A(z\varphi) = (\lambda+1)z\varphi$.
(2) *Assume $H_u^2\varphi = \gamma\varphi$.*
If $(\varphi|1) = 0$ and $(\varphi|zu) = 0$, then $\varphi = z\psi$ with $H_u^2(\psi) = \gamma\psi$.
If $(z\varphi|H_u^2(1)) = 0$ and $(\varphi|u) = 0$, then $H_u^2(z\varphi) = \gamma z\varphi$.

Lemma 7 is a straightforward consequence of the following basic identities :

$$\left\{\begin{array}{l} A(zh) = zA(h) + zh - \dfrac{1}{c}(zh|H_u^2(1))\ , \\ (A(h)|1) = -\dfrac{1}{c}(h|H_u^2(1))\ , \\ H_u^2(zh) = zH_u^2(h) + (zh|H_u^2(1)) - (h|u)zu\ . \end{array}\right. \tag{38}$$

The second lemma specifies the action of H_u on eigenfunctions of A.

Lemma 8. *Assume $A\varphi = \lambda\varphi$ and $H_u^2\varphi = \gamma\varphi$. Then*

$$AH_u\varphi = -\left(\lambda + \frac{\omega+\gamma}{c}\right) H_u\varphi.$$

If, moreover, $(\varphi|1) \neq 0$, then $\gamma = -c\lambda$ and

$$AH_u\varphi = -\frac{\omega}{c} H_u\varphi \ .$$

The first part of Lemma 8 is a simple consequence of equation (36). The second part follows from the second identity in (38), which yields

$$\lambda = -\frac{(\varphi|H_u^2(1))}{c(\varphi|1)} = -\frac{(H_u^2\varphi|1)}{c(\varphi|1)} = -\frac{\gamma}{c} \ .$$

Now we gather the important facts deduced from the above two lemmas.

Lemma 9. *The following properties hold.*

(1) $H_u(E_{\lambda,\gamma}) = E_{-(\lambda+\frac{\omega+\gamma}{c}),\gamma}$.
(2) *If* $E_{\lambda,\gamma} \not\subset 1^\perp$ *then* $\gamma = -c\lambda$.
(3) *If* $\lambda \neq 1 - \frac{\omega}{c}$ *and* $\gamma \neq -c\lambda$, *then* $E_{\lambda,\gamma} \subset zE_{\lambda-1,\gamma}$.
(4) *If* $\lambda \neq 1 - \frac{\omega}{c}$ *and* $\dim(E_{\lambda,\gamma}) \geq 2$, *then* $(\lambda - 1, \gamma) \in \Sigma$.

Proof. Lemma 8 gives that $H_u(E_{\lambda,\gamma}) \subset E_{-(\lambda+\frac{\omega+\gamma}{c}),\gamma}$. For the converse inclusion, we use the fact that, since $\ker(H_u^2 - \gamma) \subset \mathrm{Im} H_u^2$ for $\gamma > 0$, any $\varphi \in E_{-(\lambda+\frac{\omega+\gamma}{c}),\gamma}$ may be written as $\varphi = H_u(\psi)$ with $\psi \in \mathrm{Im}(H_u)$. Since $H_u^2(\varphi) = \gamma\varphi$ and H_u is one to one on $\mathrm{Im} H_u$, we get $H_u(\varphi) = \gamma\psi$ so that $A(\psi) = \frac{1}{\gamma} A(H_u(\varphi))$. We then use Equation (36) to get $A(\psi) = \lambda\psi$.

The second assertion is a direct consequence of Lemma 8.

Let us prove the third assertion. Assume $\gamma \neq -c\lambda$. Given $\varphi \in E_{\lambda,\gamma}$, assertion 2 gives $(\varphi|1) = 0$, and Lemma 7 yields $\varphi = z\psi$ with $A\psi = (\lambda - 1)\psi$. On the other hand, $(\varphi|zu) = (z\psi|zu) = (\psi|u) = 0$ since $u \in \ker(A - \frac{\omega}{c})$ and $\lambda - 1 \neq \frac{-\omega}{c}$. Hence, by Lemma 7, we have $H_u^2(\psi) = \gamma\psi$ as it is expected.

The proof of the fourth assertion is a modification of the latter, based on the following observation : if $\dim(E_{\lambda,\gamma}) \geq 2$, then $E_{\lambda,\gamma} \cap 1^\perp \neq \{0\}$. The rest of the proof is unchanged.

□

A consequence is the following description of the joint spectrum.

Lemma 10. *Given $\gamma > 0$, define $\Sigma_\gamma = \{\lambda \in \mathbb{R} \ : \ (\lambda, \gamma) \in \Sigma\}$. If Σ_γ is not empty, then there exists a nonnegative integer ℓ such that one of the following situations occurs:*

(1) *Either* $\gamma = \omega - (\ell + 2)c$ *and*

$$\Sigma_\gamma = \left\{1 - \frac{\omega}{c} + j \ , \ j = 0, \cdots, \ell\right\}$$

with the following equalities,

$$E_{-\frac{\omega}{c}+\ell+1,\omega-(\ell+2)c}=zE_{-\frac{\omega}{c}+\ell,\omega-(\ell+2)c}=\cdots=z^\ell E_{1-\frac{\omega}{c},\omega-(\ell+2)c}\ .$$

(2) *Or* $\gamma=\omega+\ell c$ *and*

$$\Sigma_\gamma=\left\{-\frac{\omega}{c}-j\ ,\ j=0,\cdots,\ell\right\}$$

with the following equalities,

$$E_{-\frac{\omega}{c},\omega+\ell c}=zE_{-\frac{\omega}{c}-1,\omega+\ell c}=\cdots=z^\ell E_{-\frac{\omega}{c}-\ell,\omega+\ell c}$$

each of the spaces being of dimension 1*.*

Proof. By the third assertion of Lemma 9, if $(\lambda,\gamma)\in\Sigma$, then

(1) either $\lambda+\frac{\omega}{c}$ is an integer ≥ 1,
(2) or $\lambda+\frac{\gamma}{c}$ is an integer ≥ 0 .

Indeed, otherwise there would exist an infinite sequence of non trivial eigenspaces

$$E_{\lambda,\gamma}\subset zE_{\lambda-1,\gamma}\subset\cdots\subset z^jE_{\lambda-j,\gamma}\subset\dots$$

since for any $j\neq 0$, $\lambda-j\neq 1-\frac{\omega}{c}$ and $\gamma\neq -c(\lambda-j)$. This would contradict the boundedness of A from below.

Applying assertion 1 of Lemma 9, these constraints also apply to the pair (λ',γ) with

$$\lambda'=-\lambda-\frac{\gamma+\omega}{c}\ .$$

This implies

(1) or $\lambda+\frac{\gamma}{c}$ is an integer ≤ -1 ,
(2) either $\lambda+\frac{\omega}{c}$ is an integer ≤ 0 .

In other words, there exists some nonnegative integer ℓ such that

(1) either $\gamma=\omega-(\ell+2)c$ and

$$\Sigma_\gamma\subset\left\{1-\frac{\omega}{c}+j\ ,\ j=0,\cdots,\ell\right\},$$

(2) or $\gamma=\omega+\ell c$ and

$$\Sigma_\gamma\subset\left\{-\frac{\omega}{c}-j\ ,\ j=0,\cdots,\ell\right\}\ .$$

Assume now that, say $\gamma=\omega-(\ell+2)c$. Applying assertion 3 of Lemma 9, we obtain, for some $k\in\{0,\cdots,\ell\}$,

$$\{0\}\neq E_{-\frac{\omega}{c}+k+1,\omega-(\ell+2)c}\subset zE_{-\frac{\omega}{c}+k,\omega-(\ell+2)c}\subset\cdots\subset z^kE_{1-\frac{\omega}{c},\omega-(\ell+2)c}\ .$$

Applying assertion 1 of Lemma 9, and again assertion 3, we also have

$$H_u(E_{1-\frac{\omega}{c},\omega-(\ell+2)c})=E_{-\frac{\omega}{c}+\ell+1,\omega-(\ell+2)c}\subset\cdots\subset z^{\ell-k}E_{-\frac{\omega}{c}+k+1,\omega-(\ell+2)c}\ .$$

Consequently, we have the claimed equality by a dimension argument.

The same procedure applies to the case $\gamma=\omega+\ell c$. Moreover, by assertion 4 of Lemma 9, we know that the dimension of $E_{-\frac{\omega}{c}-\ell,\omega+\ell c}$ is at most 1, hence exactly 1, which completes the proof. $\square$

Proof. We now turn to the proof of Proposition 9 itself. We argue by contradiction and assume that $H_u^2(u)$ and u are independent so that the eigenvalue $-\frac{\omega}{c}$ of A is not simple. As a first consequence of the fourth assertion of Lemma 9, the minimal eigenvalue of A on $\mathrm{Im}(H_u)$ is necessarily simple. By Lemma 10, since $-\frac{\omega}{c}$ is an eigenvalue of multiplicity at least 2, this minimal eigenvalue is necessarily of the form $\lambda_{min} = -\frac{\omega}{c} - j$ for some positive integer j. Again, by Lemma 10, we therefore have $\ker(A + \frac{\omega}{c}) \cap \mathrm{Im} H_u = \oplus_{k\in K} E_{-\frac{\omega}{c}, \omega+kc}$ where K is a finite subset of $\{0, \dots, j\}$ containing at least j and another integer. Furthermore, all the spaces $E_{-\frac{\omega}{c},\omega+kc}$, $k \in K$, have dimension 1.

We are going to prove that K has exactly two elements. Our strategy is based on the following observation, which is a direct consequence of Lemma 7 : if $\varphi \in \ker(A + \frac{\omega}{c})$ satisfies $(z\varphi|H_u^2(1)) = 0$, then $z\varphi$ belongs to $\ker(A - 1 + \frac{\omega}{c})$. Consequently,

$$|K| = \dim\left(\ker(A + \frac{\omega}{c}) \cap \mathrm{Im}(H_u)\right) \leq 1 + \dim(\mathcal{N}) \ ,$$

where

$$\mathcal{N} := \ker(A + \frac{\omega}{c} - 1) \cap z \left(\ker(A + \frac{\omega}{c}) \cap \mathrm{Im}(H_u)\right)$$

and, if we prove that $\mathcal{N}$ is at most one dimensional, we will conclude that $|K| = 2$.

As a first step, we are going to study the auxiliary space $\ker(A - 1 + \frac{\omega}{c}) \cap \mathrm{Im}(H_u)$. By Lemma 10, this space is the direct sum of spaces $E_{1-\frac{\omega}{c},\gamma}$, where γ describes a set of positive values included in $\{\omega - (\ell + 2)c, \ell = 0, 1, \cdots\}$. In view of assertion 2 of Lemma 9, elements ψ of $E_{1-\frac{\omega}{c},\gamma}$ satisfy $(\psi|1) = 0$, hence we can write $\psi = z\varphi$ with $\varphi \in \ker(A + \frac{\omega}{c})$, because of assertion 1 of Lemma 7. Moreover, using the third formula of (38), the equation $H_u^2\psi = \gamma\psi$ reads $(z\varphi|H_u(u)) = 0$ and

$$H_u^2\varphi = \gamma\varphi + (\varphi|u)\, u \ , \ . \tag{39}$$

hence $\varphi \in \mathrm{Im}(H_u) \cap \ker(A + \frac{\omega}{c})$. Let us compute the characteristic polynomial of the eigenvalue problem (39) on $\mathrm{Im}(H_u) \cap \ker(A + \frac{\omega}{c})$. Let $\{\varphi_k\}_{k\in K}$ be an orthonormal basis of $\ker(A + \frac{\omega}{c}) \cap \mathrm{Im} H_u = \oplus_{k\in K} E_{-\frac{\omega}{c},\omega+kc}$, with $\varphi_k \in E_{-\frac{\omega}{c},\omega+kc}$ for any $k \in K$. We write

$$\varphi = \sum_{k\in K} \alpha_k\varphi_k, \ u = \sum_{k\in K} \beta_k\varphi_k$$

Computing both sides of (39) in coordinates, we get

$$\sum_{k\in K} \alpha_k(\omega + ck)\varphi_k = \sum_{k\in K} (\gamma\alpha_k + \beta_k(\sum_{k'\in K} \alpha_{k'}\overline{\beta_{k'}}))\varphi_k$$

so that the α_k's have to satisfy the following system

$$\alpha_k(\omega + ck - \gamma) = \beta_k \sum_{k'\in K} \alpha_{k'}\overline{\beta_{k'}} \ .$$

The characteristic polynomial is the determinant of this system, namely

$$P(\gamma) = \prod_{k\in K}(\omega + kc - \gamma)\left(1 - \sum_{k\in K}\frac{|\beta_k|^2}{\omega + kc - \gamma}\right) . \tag{40}$$

Plugging the additional information $\gamma = \omega - (\ell + 2)c$ for some nonnegative integer ℓ, the equation is then equivalent to

$$\sum_{k\in K}\frac{|\beta_k|^2}{(k+\ell+2)c} = 1 , \tag{41}$$

which admits a unique simple solution in ℓ if $c > 0$, and no solution if $c < 0$. Hence $\ker(A - 1 + \frac{\omega}{c}) \cap \mathrm{Im}(H_u)$ is $\{0\}$ if $c < 0$, and is at most one dimensional if $c > 0$.

Next we distinguish two cases.

First case: $1 \notin \mathrm{Im}H_u$. Then the kernel of H_u is bL^2_+, where b is a finite Blaschke product with $b(0) \neq 0$. We infer that

$$z\mathrm{Im}(H_u) \cap \ker H_u = \{0\}.$$

Indeed, if $zH_u(\varphi) = bh$, then h is divisible by z and thus $H_u(\varphi) \in bL^2_+ = \ker H_u$, hence $H_u(\varphi) = 0$. Now we consider the orthogonal projection onto $\mathrm{Im}H_u$ restricted to $\mathcal{N}$. The kernel of this linear mapping is contained into $z\mathrm{Im}(H_u) \cap \ker H_u$, therefore this mapping is one to one. Since its image is contained into $\ker(A - 1 + \frac{\omega}{c}) \cap \mathrm{Im}(H_u)$, which is at most one dimensional, $\mathcal{N}$ is at most one dimensional. We conclude that $|K| = 2$. We notice that, in this case, we have proved that $\ker(A - 1 + \frac{\omega}{c}) \cap \mathrm{Im}(H_u)$ is exactly one -dimensional.

Second case: $1 \in \mathrm{Im}H_u$. In this case, we shall determine $\ker(A-1+\frac{\omega}{c})$ itself. Let us make some preliminary remarks. Recall from Proposition 3 that the solution $w \in \mathrm{Im}H_u$ of

$$H_u(w) = 1$$

satisfies $zw = Cb$ for some constant C where b is a Blaschke product of degree N, the Beurling generator of $\ker H_u$. From (36),

$$A(w) + \frac{\omega}{c}w = 0$$

or, since $\overline{u}w$ is holomorphic by Lemma 2,

$$Dw + \frac{\omega}{c}w = \frac{1}{c}|u|^2 w \ , \ Db + \left(\frac{\omega}{c} - 1\right) b = \frac{1}{c}|u|^2 b$$

whence, again by Rouché 's theorem,

$$Q = (N-1)c + \omega. \tag{42}$$

Observe that the above equation on b means that

$$b \in \ker(A - 1 + \frac{\omega}{c}) \ .$$

Moreover, $\ker(A-1+\frac{\omega}{c})\cap \ker H_u$ consists of functions bh satisfying

$$D(bh)-\frac{1}{c}|u|^2bh+\left(\frac{\omega}{c}-1\right)bh=0\ ,$$

or $Dh=0$. Hence

$$\ker(A-1+\frac{\omega}{c})\cap \ker H_u=\mathbb{C}b\ .$$

It remains to describe $\ker(A-1+\frac{\omega}{c})\cap \mathrm{Im}(H_u)$. We already know that this space is $\{0\}$ if $c<0$. To study the case $c>0$, we return to equation (39). We observe that $\varphi=w$ is a solution of this equation with $\gamma=0$, since $H_u^2(w)=u=(w|u)u$. Moreover, the characteristic polynomial $P(\gamma)$ given by (40) admits a unique zero in the interval $(-\infty,\min_{k\in K}(\omega+kc))$. Since this interval contains all the values $\omega-(\ell+2)c\ ,\ \ell=0,1,\cdots$, and 0 — indeed $\omega+kc, k\in K$, is an eigenvalue of H_u^2 on $\mathrm{Im}(H_u)$, hence is positive — we conclude that

$$\ker(A-1+\frac{\omega}{c})\cap \mathrm{Im}(H_u)=\{0\}\ .$$

Therefore $\ker(A-1+\frac{\omega}{c})=\mathbb{C}b$, so that $\mathcal{N}$ is at most one dimensional and $|K|=2$.

We can finally write

$$\ker(A+\frac{\omega}{c})\cap \mathrm{Im}H_u=E_{-\frac{\omega}{c},\omega+jc}\oplus E_{-\frac{\omega}{c},\omega+kc}$$

with $0\le k<j$.

As a final step, we are going to get a contradiction implied by this two-dimensionality.

We first consider the case when $1\in \mathrm{Im}(H_u)$. Let us apply the Cayley-Hamilton theorem to H_u^2 on the two-dimensional space $\ker(A+\frac{\omega}{c})$. We obtain

$$H_u^4(u)-(2\omega+(j+k)c)H_u^2(u)+(\omega+jc)(\omega+kc)u=0\ .$$

Since $1\in \mathrm{Im}(H_u)$, this implies

$$H_u^4(1)-(2\omega+(j+k)c)H_u^2(1)+(\omega+jc)(\omega+kc)=0\ ,$$

and, taking the scalar product with 1,

$$J_4-(2\omega+(j+k)c)Q+(\omega+jc)(\omega+kc)=0\ .$$

Using that $Q=(N-1)c+\omega$ by (42), we get

$$\begin{aligned} J_4-Q^2 &= (2\omega+(j+k)c-(N-1)c-\omega)((N-1)c+\omega)-(\omega+jc)(\omega+kc)\\ &= -c^2(N-1+jk)<0\ . \end{aligned}$$

This fact is in contradiction with the Cauchy-Schwarz inequality,

$$Q^2=|(H_u^2(1)|1)|^2\le \|H_u^2(1)\|^2=(H_u^4(1)|1)=J_4\ .$$

It remains to consider the case $1 \notin \mathrm{Im}(H_u)$. Again, we are going to contradict the Cauchy-Schwarz inequality. First, we use the Cayley-Hamilton theorem as before,

$$H_u^4(1) - (2\omega + (j+k)c)H_u^2(1) + (\omega + jc)(\omega + kc)P_u(1) = 0 \ ,$$

which yields to

$$J_4 - (2\omega + (j+k)c)Q + (\omega + jc)(\omega + kc)(1 - S) = 0$$

and

$$J_4(1-S) - Q^2 = -\frac{(J_4 - Q(\omega + jc))(J_4 - Q(\omega + kc))}{(\omega + jc)(\omega + kc)} \ . \tag{43}$$

The Cauchy–Schwarz inequality

$$Q^2 = |(H_u^2(1)|P_u(1))|^2 \le \|H_u^2(1)\|^2 \|P_u(1)\|^2 = J_4(1-S)$$

implies that the left hand side of (43) is nonnegative. Therefore, remembering that $\omega + jc$ and $\omega + kc$ are positive as eigenvalues of H_u^2 on $\mathrm{Im}(H_u)$, we shall obtain a contradiction if we show that

$$J_4 \ > \ Q(\omega + jc) \ . \tag{44}$$

Let us prove (44). Recall that $c > 0$, since $Q = Nc$. Apply Lemma 10. If $\gamma > 0$ is an eigenvalue of H_u^2, either $\gamma = \omega + \ell c$ with $\ell \ge 0$, and $E_{-\frac{\omega}{c},\gamma} \ne \{0\}$, and this implies $\ell \in \{j,k\}$; or $\gamma = \omega - (\ell+2)c$, $\ell \ge 0$, and $E_{1-\frac{\omega}{c},\gamma} \ne \{0\}$. In this case, we have already seen that $\ker(A - 1 + \frac{\omega}{c}) \cap \mathrm{Im}(H_u)$ is one dimensional, which means that ℓ is uniquely determined and $E_{1-\frac{\omega}{c},\gamma}$ is one-dimensional. We infer the following decomposition, where all the spaces $E_{\lambda,\gamma}$ are one–dimensional,

$$\begin{aligned}
\mathrm{Im}(H_u) &= E_1 \oplus E_2 \oplus E_3 \ , \\
E_1 &= \oplus_{j'=0}^{j} E_{-\frac{\omega}{c} - j', \omega + jc} \ , \\
E_2 &= \oplus_{k'=0}^{k} E_{-\frac{\omega}{c} - k', \omega + kc} \ , \\
E_3 &= \oplus_{\ell'=0}^{\ell} E_{1-\frac{\omega}{c} + \ell', \omega - (\ell+2)c} \ .
\end{aligned}$$

Consequently, $N = j + k + \ell + 3$ and

$$\begin{aligned}
\mathrm{Tr}(H_u^2) &= (j+1)(\omega + jc) + (k+1)(\omega + kc) + (\ell+1)(\omega - (\ell+2)c) \\
&= N\omega + c[j(j+1) + k(k+1) - (\ell+1)(\ell+2)].
\end{aligned}$$

On the other hand, $\mathrm{Tr}(H_u^2) = M + Q = M + Nc$, and, taking the scalar product of u with both sides of the soliton equation (35), we have,

$$M + \frac{\omega}{c} Q = \frac{1}{c}(2J_4 - Q^2) \ .$$

Using the identity $Q = Nc$, we infer

$$2J_4 = Mc + N\omega c + N^2 c^2 \ ,$$

and, using the above expression of M provided by the trace of H_u^2,

$$2J_4 = 2N\omega c + c^2(N^2 + j(j+1) + k(k+1) - (\ell+1)(\ell+2) - N) \ .$$

Consequently,

$$\begin{aligned}2(J_4 - Q(\omega + jc)) &= 2J_4 - 2N\omega c - 2Njc^2\\ &= c^2(N^2 + j(j+1) + k(k+1) - (\ell+1)(\ell+2) - N(2j+1))\\ &= 2c^2(k+1)(k+\ell+2) > 0\end{aligned}$$

as can be shown by a straightforward calculation. This proves (44) and yields the contradiction, completing the proof of Theorem 9.1. □

10. Appendices

10.1. **Appendix 1: The Brezis Gallouët estimate.**

We recall a simple proof of the estimate

$$\|u\|_{L^\infty} \le C_s \|u\|_{H^{1/2}} \left[\log\left(2 + \frac{\|u\|_{H^s}}{\|u\|_{H^{1/2}}}\right)\right]^{\frac{1}{2}} .$$

By Fourier expansion, one has, for any $N \in \mathbb{N}$

$$\begin{aligned} & \|u\|_{L^\infty} \le \sum |\hat{u}(k)| \\ = & \sum_{|k|\le N} (1+|k|)^{1/2} \frac{|\hat{u}(k)|}{(1+|k|)^{1/2}} + \sum_{|k|\ge N+1} (1+|k|)^s \frac{|\hat{u}(k)|}{(1+|k|)^s} \\ \le & \|u\|_{H^{1/2}} \times \left(\sum_{|k|\le N} \frac{1}{1+|k|}\right)^{1/2} + \|u\|_{H^s} \times \left(\sum_{|k|\ge N+1} \frac{1}{(1+|k|)^{2s}}\right)^{1/2} \\ \le & C\left(\|u\|_{H^{1/2}} \log(N+1)^{1/2} + \|u\|_{H^s} N^{-s+1/2}\right). \end{aligned}$$

The result follows by taking the minimum over N.

10.2. **Appendix 2: A Trudinger-type estimate.**

Let us prove the estimate

$$\forall p < \infty\,,\ \|u\|_{L^p} \le C\sqrt{p}\,\|u\|_{H^{1/2}} \ . \tag{45}$$

It follows from a Marcinkiewicz type argument. Assume $\|u\|_{H^{1/2}} = 1$. Write, for any $p > 2$,

$$\|u\|_{L^p}^p = p\int_0^\infty t^{p-1}\sigma(\{x,\ |u(x)| \ge t\})dt$$

and decompose $u = u_{>\lambda} + u_{<\lambda}$ where $u_{<\lambda} = \sum_{|k|\le\lambda} \hat{u}(k)e^{ik\theta}$. Choose $\lambda = \lambda_t$ so that $\|u_{<\lambda}\|_\infty \le t/2$. More precisely, since

$$\begin{aligned} \|u_{<\lambda}\|_\infty &\le \sum_{|k|\le\lambda} |\hat{u}(k)| \\ &\lesssim \left(\sum_{|k|\le\lambda} (|k|^2+1)^{1/2}|\hat{u}(k)|^2\right)^{1/2} \times [\log(\lambda+1)]^{1/2} \\ &\lesssim \|u\|_{H^{1/2}}[\log(\lambda+1)]^{1/2} = c[\log(\lambda+1)]^{1/2}, \end{aligned}$$

we can choose λ so that $c[\log(\lambda+1)]^{1/2} = \frac{t}{2}$. With this choice, we get

$$\begin{aligned} \|u\|_{L^p}^p &\le p\int_0^\infty t^{p-1}\sigma(\{x,\ |u_{>\lambda_t}(x)| \ge t/2\})dt \\ &\le p\int_0^\infty t^{p-3}\|u_{>\lambda_t}\|_2^2 dt \le p\int_0^\infty t^{p-3}\sum_{|k|\ge\lambda_t}|\hat{u}(k)|^2 dt \\ &\le p\sum_k \left(\int_0^{2\log(|k|+1)^{1/2}} t^{p-3}dt\right)|\hat{u}(k)|^2 \\ &\le \frac{p}{p-2}\sum_k (\log(|k|+1))^{(p-2)/2}|\hat{u}(k)|^2. \end{aligned}$$

Eventually, we use that $(\log(|k|+1))^\ell \lesssim \ell!(|k|+1) \lesssim \ell^\ell(|k|^2+1)^{1/2}$. It gives the expected constant proportional to $p^{1/2}$ in (45).

10.3. **Appendix 3: An elementary proof of the Kronecker Theorem.** Let $u \in BMO_+(\mathbb{S}^1)$ so that the Hankel operator H_u is well defined as a bounded operator on $L^2_+(\mathbb{S}^1)$. Since H_u is $\mathbb{C}$ -antilinear, the range of H_u is a complex vector space.

Proposition 10. *The function u belongs to $\mathcal{M}(N)$ if and only if the Hankel operator H_u has (complex) rank N. Moreover, if*

$$B(z) = \prod_{j=1}^N (1 - p_j z)$$

is the denominator of u, the image of H_u is the vector space generated by

$$\frac{1}{(1-pz)^m}$$

for $0 < |p| < 1, 1 \le m \le m_p$, or of the form

$$z^m, 0 \le m \le m_0 - 1\ ,$$

where m_p is the number of occurrences of p in the list $p_1, \cdots, p_N$.

Proof. The proof is based on the following two observations.

i) If $u \in \mathcal{M}(N)$, then $\mathrm{rk}(H_u) \leq N$.

ii) If $\mathrm{rk}(H_u) = N$, then $u \in \mathcal{M}(N)$.

Let us first prove i). If $u \in \mathcal{M}(N)$, then one can write u as a linear combination of functions of the form

$$\frac{1}{(1-pz)^m}$$

for $0 < |p| < 1, 1 \leq m \leq m_p$, or of the form

$$z^m, 0 \leq m \leq m_0 - 1 \ ,$$

which we shall associate to $p = 0$, with the following degree condition,

$$\sum_p m_p = N \ .$$

Indeed, either the denominator of u is of degree N, and this corresponds to the fact that all the p's are different from 0, and the above identity reflects the degree of the denominator ; or the denominator has degree $< N$, and then the numerator should be of degree exactly $N - 1$; therefore the decomposition of u into elementary fractions involves a polynomial function of degree $m_0 - 1 \geq 0$, and the above identity reflects the degree of the numerator $+1$. Now we recall that

$$\widehat{H_u(h)}(k) = \sum_{\ell \geq 0} \hat{u}(k+\ell)\overline{\hat{h}(\ell)} \ .$$

In view of the decomposition of u, we observe that the sequence $(\hat{u}(k))_{k\geq 0}$ is a linear combination of the following sequences,

$$k^{m-1}p^k, 1 \leq m \leq m_p \ ,$$

for $p \neq 0$, and

$$\delta_{km}, 0 \leq m \leq m_0 - 1 \ ,$$

for $p = 0$. This implies that all the sequences $(\widehat{H_u(h)}(k))_{k\geq 0}$ have the same property, and therefore the range of H_u is included into the space V of linear combinations of

$$\frac{1}{(1-pz)^m}, 1 \leq m \leq m_p, 0 < |p| < 1 \ ; \ z^m, 0 \leq m \leq m_0 - 1 \ .$$

This implies that $\mathrm{rk}(H_u) \leq N$.

We now proceed to the proof of property ii). We know that H_u is a symmetric operator of real rank $2N$. Restricting H_u to its range, which is a complex vector space of dimension N and is the orthogonal of $\mathrm{Ker}(H_u)$ (for both real scalar product and hermitian scalar product), we can find a real orthonormal basis of eigenvectors of H_u. Moreover, since H_u is antilinear, we observe that, if $H_u(v) = \lambda v$, then $H_u(iv) = -i\lambda v$.

Therefore we may assume that the above real orthonormal basis of $\mathrm{Im}(H_u)$ has the special form

$$v_1, iv_1, v_2, iv_2, \dots, v_N, iv_N \ ,$$

and that $H_u(v_j) = \lambda_j v_j$ with some $\lambda_j > 0$. Defining $w_j := \sqrt{\lambda_j} v_j$, we obtain the following expression for H_u,

$$H_u(h) = \sum_{j=1}^{N} (w_j|h)_{L^2}\, w_j \ ,$$

or equivalently,

$$\hat{u}(k+\ell) = \sum_{j=1}^{N} \hat{w}_j(k)\hat{w}_j(\ell) \ ,$$

for all $k \geq 0, \ell \geq 0$. Now the matrix $(\hat{w}_j(\ell))_{1\leq j\leq N, 0\leq \ell\leq N}$ has rank at most N, therefore there exists $(c_0, c_1, \dots, c_N) \neq (0, \dots, 0)$ in $\mathbb{C}^{N+1}$ such that

$$\sum_{\ell=0}^{N} c_\ell\, \hat{w}_j(\ell) = 0$$

for every $j = 1, \dots, N$. This implies that

$$\sum_{\ell=0}^{N} c_\ell\, \hat{u}(k+\ell) = 0 \ ,$$

for every $k \geq 0$. We then introduce the polynomial

$$P(X) = \sum_{\ell=0}^{N} c_\ell X^\ell \ .$$

Let

$$\mathcal{P} = \{p \in \mathbb{C}, P(p) = 0\}$$

and $m_p \geq 1$ denotes the multiplicity of $p \in \mathcal{P}$. Then the theory of linear recurrent sequences implies that the sequence $(\hat{u}(k))_{k\geq 0}$ is a linear combination of the following sequences,

$$k^{m-1}p^k, 1 \leq m \leq m_p \ ,$$

for $p \neq 0$, and

$$\delta_{km}, 0 \leq m \leq m_0 - 1 \ ,$$

for $p = 0$. In other words, u is a linear combination of the following functions,

$$\frac{1}{(1-pz)^m}\ , \ 1 \leq m \leq m_p\, , \ 0 < |p| < 1 \ ; \ z^m, 0 \leq m \leq m_0 - 1 \ .$$

Since $\sum_p m_p \leq N$, this implies that $u \in \mathcal{M}(N')$ for some $N' \leq N$. However, if $N' < N$, assertion i) would imply $\mathrm{rk}(H_u) \leq N'$, which contradicts the assumption. Therefore $N' = N$, and ii) is proved.

Finally, in view of ii), i) is strengthened into
i)' If $u \in \mathcal{M}(N)$, then $\mathrm{rk}(H_u) = N$.

Moreover, the inclusion of the range of H_u into the space V becomes an equality, which is the claim.

This completes the proof. □

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